• **PROBLEM 13-96**

Determine the stability of both discrete systems

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad (1)$$

and

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.4 & 1 \\ 0 & 0.3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \qquad (2)$$

Solution: From $\det|\lambda I - A| = 0$ we compute the eigenvalues of the matrix A for the first system.

$$(\lambda - 1)^2 - 6 = 0$$

$$\lambda = 1 + \sqrt{6}$$

The system is unstable since eigenvalues are greater than unity.

For the second system we have

$$\det \begin{pmatrix} \lambda-0.4 & -1 \\ 0 & \lambda-0.3 \end{pmatrix} = 0$$

$$(\lambda - 0.4)(\lambda - 0.3) = 0$$

$$\lambda_1 = 0.4$$

$$\lambda_2 = 0.3$$

The system (2) is stable since

$$|\lambda_1| < 1 \quad \text{and} \quad |\lambda_2| < 1.$$

• **PROBLEM 13-97**

Show that the system described by the equation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

where k is a discrete parameter, is unstable. Find a scalar function H such that

$$u(k) = -Hy(k)$$

will stabilize the system.

Solution: The eigenvalues of the matrix A

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix}$$

are $\lambda_1 = -1$, $\lambda_2 = -4$ so the open-loop system is unstable.

In order to stabilize the system we seek a scalar function H, such that taking into account the output feedback

$$u(k) = -Hy(k)$$

the eigenvalues of $\overline{A}$, where $\overline{A}$ is a closed-loop matrix are less than unity in magnitude.

$$\overline{A} = A - BHC = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} H[1\ \ 0]$$

$$= \begin{bmatrix} 0 & 1 \\ -4-H & -5 \end{bmatrix}$$

The characteristic polynomial of $\overline{A}$ is

$$\det |\lambda I - \overline{A}| = \det \begin{vmatrix} \lambda & -1 \\ 4 + H & \lambda + 5 \end{vmatrix} = 0$$

$$\lambda^2 + 5\lambda + 4 + H = 0$$

$$\lambda_1 = -\frac{5}{2} + \sqrt{25-4(4+H)}$$

$$\lambda_2 = -\frac{5}{2} - \sqrt{25-4(4+H)}$$

We notice that there is no value of H that can make both roots of the polynomial less than unity in magnitude, hence the system is not stable and cannot be stabilized using output feedback.

Check the stability of the following system

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = 2.5x_1(k) + x_2(k) + u(k)$$

$$y(k) = x_1(k)$$

If the system is unstable, use the output feedback, $u(k) = -Hy(k)$ to stabilize the system, determine the range of H.

Solution: We rewrite the equations in the vector notation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k)$$

From $\det|\lambda I - A| = 0$ we calculate the eigenvalues of the matrix A

$$\lambda^2 - \lambda - 2.5 = 0$$

$$\lambda_1 = \frac{1}{2} + \sqrt{\frac{11}{4}} = 2.16$$

$$\lambda_2 = \frac{1}{2} - \sqrt{\frac{11}{4}} = -1.16$$

The open-loop system is unstable. We shall try to stabilize the system using output feedback

$$u(k) = -Hy(k)$$

and determining the range of H. The closed-loop matrix $\bar{A}$ is

$$\bar{A} = A - BHC = \begin{bmatrix} 0 & 1 \\ 2.5 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} H[1 \quad 0] = \begin{bmatrix} 0 & 1 \\ 2.5 - H & 1 \end{bmatrix}$$

The eigenvalues of $\bar{A}$ are

$$\det|\lambda I - \bar{A}| = 0$$

$$\det \begin{vmatrix} \lambda & -1 \\ H - 2.5 & \lambda - 1 \end{vmatrix} = 0$$

$$\lambda^2 - \lambda + (H - 2.5) = 0$$

$$\lambda_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4(H - 2.5)} = \frac{1}{2} + \frac{1}{2}\sqrt{11 - 4H}$$

$$\lambda_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4(H - 2.5)} = \frac{1}{2} - \frac{1}{2}\sqrt{11 - 4H}$$

If we assume that λ_1, λ_2 are real we get the following range of H

$$2.5 \le H \le 2.75$$

for which the system is stable. If the roots are complex we have

$$2.75 \le H \le 3.5$$

We conclude that the system is stable when $2.5 \le H \le 3.5$

• PROBLEM 13-99

The discrete-time system is shown in Fig. (1).

The open-loop transfer function of the system is

$$G(s) = \frac{5}{s(s + 1)}$$

Examine the stability of the system.

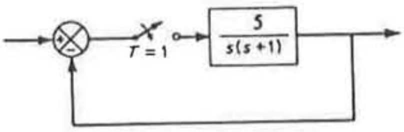


Fig. 1

Solution: We find the z - transformation of G(s).

$$G(s) = \frac{5}{s(s + 1)} = 5\left(\frac{1}{s} - \frac{1}{s + 1}\right)$$

and

$$g(t) = 5(u(t) - e^{-t}).$$

Substituting $t = kT$, where $T = 1$ we obtain

$$g(kT) = g(k) = 5(u(k) - e^{-k}).$$

Then

$$G(z) = \sum_{k=0}^{\infty} g(k)z^{-k} = \sum_{k=0}^{\infty} 5(u(k) - e^{-k})z^{-k} = 5\left(\frac{z}{z - 1} - \frac{z}{z - e^{-1}}\right)$$

Thus

$$G(z) = \frac{5(1 - e^{-1})z}{(z - 1)(z - e^{-1})}$$

The characteristic equation of the system is

$$1 + G(z) = 0$$

or

$$5(1 - e^{-1})z + (z - 1)(z - e^{-1}) = 0$$

$$e^{-1} = 0.368$$

$$z^2 - 0.368z - z + 0.368 + 5z - 1.84z = 0$$

$$z^2 + 1.792z + 0.368 = 0$$

$$z_1 = -0.236$$

$$z_2 = -1.555$$

We see that root z_2 of the characteristic equation has a magnitude greater than unity, thus the system is unstable. It can be easily shown that for the system shown in Fig.(1) with the transfer function

$$G(s) = \frac{K}{s(s + 1)}$$

the condition of stability is

$$0 < K < 4.32.$$

In case of a characteristic equation of higher degree, where the exact solution is difficult or impossible to find, we use the following transformation of z

$$z = \frac{r + 1}{r - 1}.$$

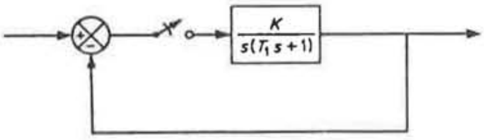
To check the stability of resulting equation one applies the Routh criterion. In the above example we obtain

$$\left(\frac{r + 1}{r - 1}\right)^2 + 1.792\left(\frac{r + 1}{r - 1}\right) + 0.368 = 0$$

$$3.16r^2 + 1.264r - 1.16 = 0$$

All the coefficients of the equation must be non-zero and have the same sign for the system to be stable, this is the necessary but not sufficient condition. We conclude that the system is not stable. This corresponds to the result obtained previously.

Determine the stability of a closed-loop discrete-time system shown below



Solution: The gain of the system is

$$G(s) = \frac{K}{s(T_1 s + 1)}$$

To find the z - transform of G(s) note that

$$G(s) = \frac{K}{s(s + \frac{1}{T_1})T_1} = K\frac{\frac{1}{T_1}}{s(s + \frac{1}{T_1})}$$

The z transform of

$$\frac{a}{s(s + a)} \text{ is } \frac{(1 - e^{-aT})z}{(z - 1)(z - e^{-aT})}$$

thus

$$G(z) = \frac{K\left(1 - e^{-\frac{T}{T_1}}\right)z}{(z-1)(z-e^{-\frac{T}{T_1}})}$$

The characteristic equation is

$$1 + G(z) = 0$$

or

$$1 + \frac{K\left(1 - e^{-\frac{T}{T_1}}\right)z}{(z - 1)(z - e^{-\frac{T}{T_1}})} = 0$$

After some calculations we get

$$z^2 + \left[K\left(1 - e^{-\frac{T}{T_1}}\right) - \left(1 + e^{-\frac{T}{T_1}}\right)\right]z + e^{-\frac{T}{T_1}} = 0$$

In order to use the Routh criterion we transform z,

$$z = \frac{r + 1}{r - 1}$$

The characteristic equation becomes

$$r^2\left[K\left(1 - e^{-\frac{T}{T_1}}\right)\right] + r \cdot 2\left(1 - e^{-\frac{T}{T_1}}\right) + 2\left(1 + e^{-\frac{T}{T_1}}\right)$$
$$- K\left(1 - e^{-\frac{T}{T_1}}\right) = 0$$

The Routh tabulation is

$$\begin{array}{lll} r^2 & K\left(1 - e^{-\frac{T}{T_1}}\right) & 2\left(1 + e^{-\frac{T}{T_1}}\right) - K\left(1 - e^{-\frac{T}{T_1}}\right) \\ r^1 & 2\left(1 - e^{-\frac{T}{T_1}}\right) & 0 \\ r^0 & 2\left(1 + e^{-\frac{T}{T_1}}\right) - K\left(1 - e^{-\frac{T}{T_1}}\right) & \end{array}$$

Note that $1 - e^{-\frac{T}{T_1}} > 0$ and the system is stable if and only if $K > 0$

$$\frac{1 + e^{-\frac{T}{T_1}}}{1 - e^{-\frac{T}{T_1}}} > \frac{K}{2}$$

Since

$$\frac{1 + e^{-\frac{T}{T_1}}}{1 - e^{-\frac{T}{T_1}}} = \coth\left(\frac{T}{2T_1}\right)$$

We conclude that the system is stable when

$$0 < K < 2 \coth\left(\frac{T}{2T_1}\right)$$

PHASE PLANE

• **PROBLEM 13-101**

Determine the stability for the system described by the equation

$$x(k+1) = Ax(k) \tag{1}$$

where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution: Using the z-transform of both sides of Eq. 1 and solving for X(z) we obtain

$$X(z) = (zI-A)^{-1} zx(0)$$

or

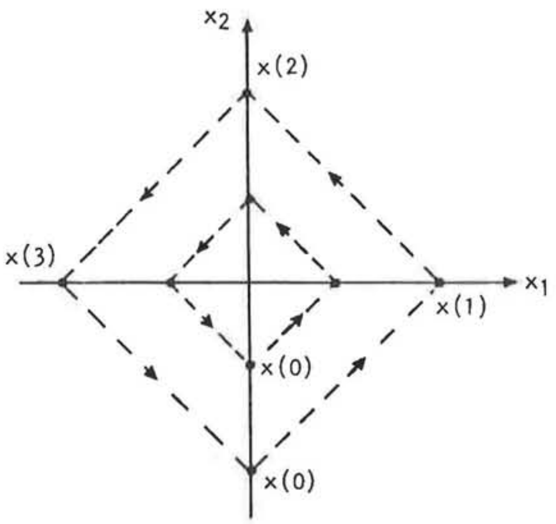
$$\begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \frac{1}{z^2+1} \begin{bmatrix} z^2 & -z \\ z & z^2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{2}$$

The functions $x_1(k)$ and $x_2(k)$ can be found by taking the inverse z-transform of Eq. 2. We obtain

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} \cos\frac{k\pi}{2} & -\sin\frac{k\pi}{2} \\ \sin\frac{k\pi}{2} & \cos\frac{k\pi}{2} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

We will find the state trajectories of $x_1(k)$ and $x_2(k)$ for two arbitrary initial states x(0).

$$\begin{aligned} x_1(k) &= \cos\frac{k\pi}{2}\, x_1(0) - \sin\frac{k\pi}{2}\, x_2(0) \\ x_2(k) &= \sin\frac{k\pi}{2}\, x_1(0) + \cos\frac{k\pi}{2}\, x_2(0) \end{aligned} \tag{3}$$



STATE-PLANE TRAJECTORIES

From Eq. (3) we conclude that trajectories form limit cycle; $x_1(k)$ and $x_2(k)$ are periodic functions which neither grow nor decay in amplitude. Therefore the system is stable but not asymptotically stable.

We can determine the stability of the system without finding the trajectories.

The characteristic equation of the system is

$$|zI-A| = z^2 + 1 = 0$$

The characteristic equation has two roots $z_1 = j$ $z_2 = -j$ which are inside the unit circle $|z| = 1$ in the z-plane.

Therefore the system is stable.

• PROBLEM 13-102

Given the system

$$x(k+1) = Ax(k)$$

where k is a discrete independent variable and matrix A is given

$$A = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

determine the stability of the system.

Solution: Using the z-transform we obtain

$$X(z) = \begin{bmatrix} \frac{z}{z+0.5} & 0 \\ 0 & \frac{z}{z-0.5} \end{bmatrix} x(0)$$

where

$$X(z) = (zI-A)^{-1} zx(0)$$

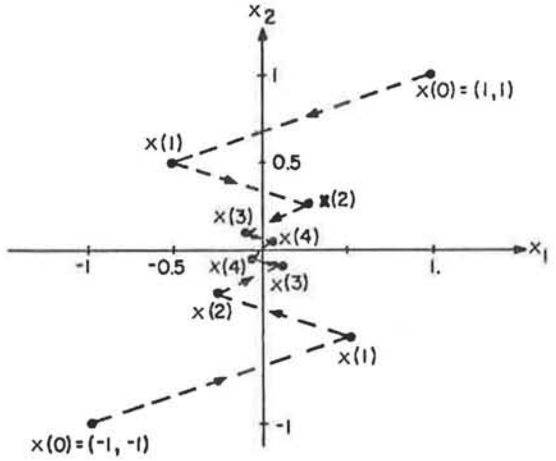
The inverse z-transform of X(z) gives

$$x(k) = \begin{bmatrix} (-0.5)^k & 0 \\ 0 & (0.5)^k \end{bmatrix} x(0)$$

For any initial state x(0) the state trajectories of x(k) will converge to the equilibrium state x = 0 as k tends to infinity.

Therefore the system is asymptotically stable in the large.

Let x(0) = [1 1]' and [-1 -1]' be initial states. The state trajectories of x(k) are shown below.



State-plane trajectories of the system

The characteristic equation of the system is

$$|zI-A| = (z-0.5)(z+0.5) = 0$$

The equation has two real roots $z_1 = -0.5 \quad z_2 = 0.5$
They both are inside the unit circle $|z| = 1$.

• **PROBLEM** 13-103

The system is described by the equation

$$x(k+1) = Ax(k)$$

where k is a discrete independent variable and A is a diagonal matrix

$$A = \begin{bmatrix} -2.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Examine the stability of the system.

Solution: From $X(z) = (zI-A)^{-1} zx(0)$

which is the solution in the z domain we obtain

$$X(z) = \begin{bmatrix} \frac{z}{z+2.5} & 0 \\ 0 & \frac{z}{z-0.5} \end{bmatrix} x(0)$$

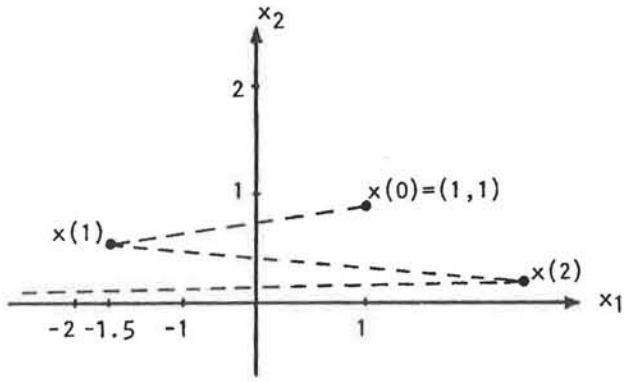
We note that the characteristic equation of the system has a root at $z = -2.5$ which is outside the unit circle.

Therefore the system is unstable.

The inverse z-transform of X(z) gives

$$x(k) = \begin{bmatrix} (-2.5)^k & 0 \\ 0 & (0.5)^k \end{bmatrix} x(0)$$

It is easy to show that $|x_1(k)| \to \infty$ as k approaches infinity.



State-plane trajectories of the system

Let the initial state (k=0) be

$x(0) = [1,1]$. We have

$$\underset{k=0}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \rightarrow \underset{k=1}{\begin{bmatrix} -2.5 \\ 0.5 \end{bmatrix}} \rightarrow \underset{k=2}{\begin{bmatrix} 6.25 \\ 0.25 \end{bmatrix}} \rightarrow \underset{k=3}{\begin{bmatrix} -15.6 \\ 0.125 \end{bmatrix}}$$

• PROBLEM 13-104

Describe the motion of a pendulum and determine its stability characteristics. Assume that the whole mass is concentrated at the end of the pendulum.

Solution: The motion of a pendulum is described by the following differential state equation

$$\ddot{\phi} = -\frac{g}{L}\sin\phi$$

We can choose the system of physical units where $\frac{g}{L} = 1$.

We have

$$\ddot{\phi} = -\sin\phi$$

Let us set

$$x_1 = \phi$$
$$x_2 = \dot{\phi}$$

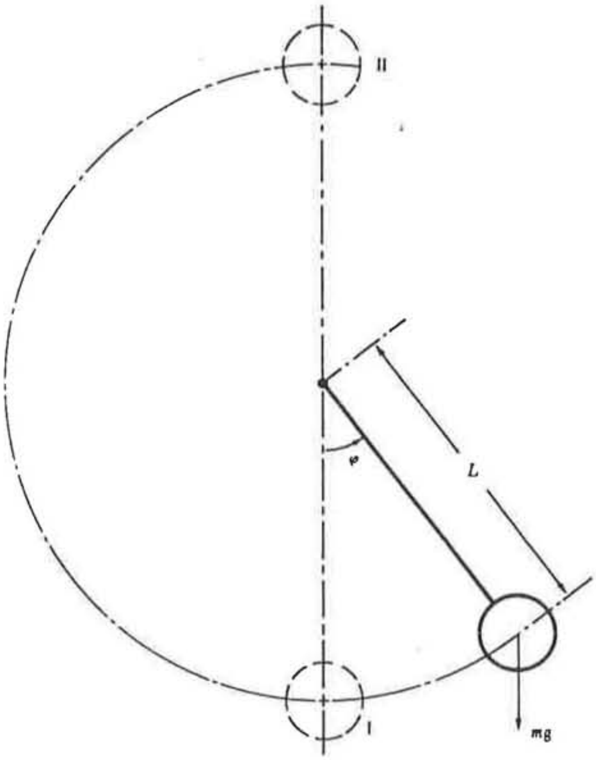


Fig. 1

Simple pendulum.

and obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin x_1 \qquad (1)$$

The equilibrium states are

$$x_I^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{II}^0 = \begin{bmatrix} \pm\pi \\ 0 \end{bmatrix}$$

as shown in Fig(1)
The Jacobian matrix of (1) is

$$A = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & 0 \end{bmatrix}$$

and for the equilibrium states is

$$A_I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad A_{II} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

From the equation $|A - \lambda I| = 0$ we calculate the eigenvalues

$$\det \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \qquad \text{which gives for}$$

x_1 , $\lambda = \pm j$ λ a vortex,

and

$$\det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \qquad \text{which gives for}$$

x_2 , $\lambda = \pm 1$ a saddle.

To determine the stability in the finite phase plane we have to find the phase trajectories, that is the relationship $x_2 = x_2(x_1)$.

From (1) we get

$$x_2 dx_2 = - \sin x_1 \, dx_1$$

and integrating

$$x_2^2 = 2 \cos x_1 + C$$

where C is a constant.

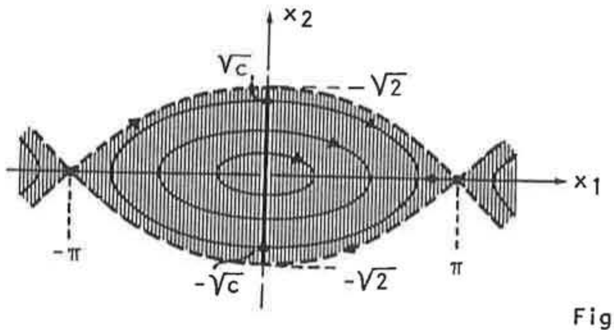


Fig. 2

phase trajectories $x_2^2 = 2 \cos x_1 + c$

Different values of C give different trajectories. To find the trajectory that passes through the saddle points at $x_1 = \pm\pi$

substitute $x = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$. We have

$0 = 2 \cos \pi + C$ thus

$x_2^2 = 2 \cos x_1 + 2$

The trajectories that pass through $x = \pm\pi$ are called separatrices, they separate the stable regions from the unstable ones. The shaded area represents the stable region.

Examine the stability of the second order regulator with nonlinear damping shown in Fig. (1).

The transfer function of the nonlinear plant is

$$G_p = \frac{1}{s^2}$$

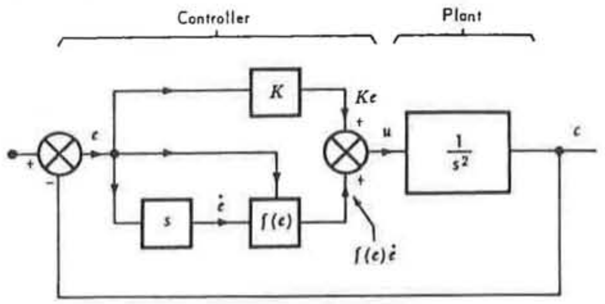


Fig. 1

Proportional-plus-nonlinear derivative control.

Solution: From the diagram we have

$$u = Ke + f(e)\dot{e}$$

where u is the control force. The amount of the derivative control depends upon f(e) - the magnitude of the error.

The following equation describes the dynamic of the system

$$u = \ddot{e}$$

To obtain the set of state equations let $e = x_1$ and $\dot{e} = x_2$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -Kx_1 - f(x_1)\, x_2$$

Let us assume for simplicity that K = 1. The equilibrium points are determined from the equation

$$0 = x_2^0$$

$$0 = -x_1^0 - f(x_1^0)x_2^0$$

Thus $x_1^0 = x_2^0 = 0$ is the only point.

The Jacobian matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -f(0) \end{bmatrix}$$

The linear behavior of the system in the close neighborhood of the origin is described by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - f(0)x_2$$

and the characteristic equation is

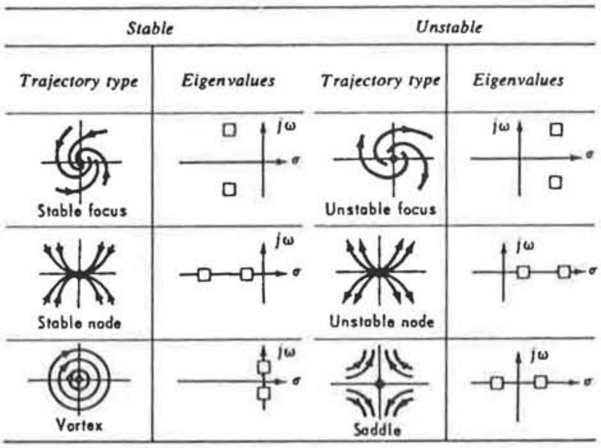
$$s[s + f(0)] + 1 = 0$$

with the eigenvalues

$$\lambda_1 = -\frac{1}{2} f(0) + \sqrt{\frac{1}{4} f^2(0) - 1}$$

$$\lambda_2 = -\frac{1}{2} f(0) - \sqrt{\frac{1}{4} f^2(0) - 1}$$

From the table below



VARIOUS TYPES OF SINGULARITIES

we obtain the following possible responses:

1. $f(0) > 2$ stable node
2. $0 < f(0) < 2$ stable focus
3. $f(0) = 0$ vortex
4. $-2 < f(0) < 0$ unstable focus
5. $f(0) < -2$ unstable node

The first two cases ensure asymptotic stability, and the third case local stability around the origin.

Next we shall investigate the response of the system subjected to large input steps, i.e., the problem of the finite and global stability of the system.

Using the phase plane we shall demonstrate that it is possible to control and shape the behavior of a nonlinear system. For this purpose let us assume the following types of derivative control:

I. $f(e) = 2$

II. $f(e) = 1$

III. $f(e) = 0$

IV. $f(e) = \frac{0.1}{|e|}$

The first three cases are linear, thus

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - f(0)x_2$$

holds throughout the entire phase plane. We get the following slope equations using the isocline method:

$$\text{I.} \quad S = -\frac{x_1 + 2x_2}{x_2} = -2 - \frac{x_1}{x_2}$$

$$\text{II.} \quad S = -1 - \frac{x_1}{x_2}$$

$$\text{III.} \quad S = -\frac{x_1}{x_2}$$

In all three cases the slope is constant for constant ratio

$$\frac{x_1}{x_2},$$

therefore the isoclines are straight lines through origin.

Fig. 2 shows the phase trajectories and Fig. 3 the corresponding time responses.

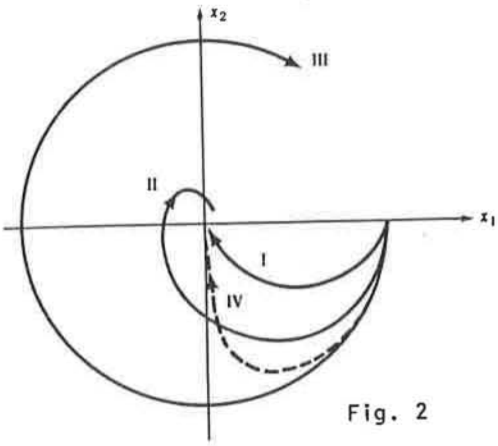


Fig. 2

Note that in Fig. 3 the time responses range from undamped to overdamped. By increasing the damping we sacrifice the response of the system.

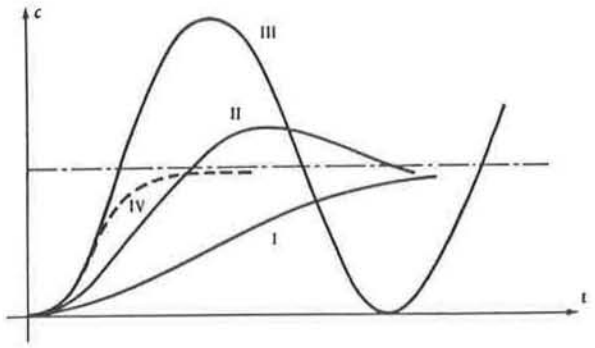


Fig. 3

To find the right combination of time response and damping let us investigate case IV:

$$f(e) = \frac{0.1}{|e|}$$

We obtain the isocline equation

$$\frac{dx_2}{dx_1} = S = -\frac{x_1 + 0.1\frac{x_2}{|x_1|}}{x_2} = -\frac{x_1}{x_2} - 0.1\frac{1}{|x_1|}$$

The phase trajectory and the corresponding time response are shown in Fig(2) and (3).

ROOT LOCUS

• **PROBLEM** 13-106

For the system of Fig. 1, where $K \geq 0$

a) Sketch the root locus for the system, giving starting and ending points, asymptote intersects and angles.
b) Using Hurwitz test, find the values of $K \geq 0$ for which the system becomes unstable in the Lyapunov sense.
c) Calculate the overall transfer function H(s). Using it, analyze results of a) and b).

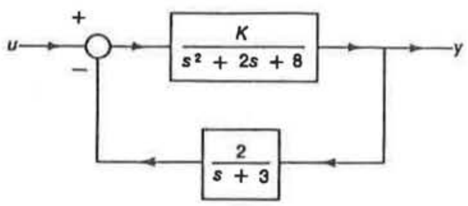


Fig. 1

Solution: We have

$$H_1 H_f = \frac{1}{(s + 3)(s^2 + 2s + 8)}$$

The poles are -3, $-1 + j\sqrt{7}$, $-1 - j\sqrt{7}$. The locus starts at the poles and ends at the zeros, all of which are at infinity. The asymptote angles are

$$-\frac{1}{3}(2k + 1)\pi \text{ or } -\frac{\pi}{3}, -\pi, -\frac{5}{3}\pi .$$

The asymptote intersection on the real axis is

$$\sigma_a = \frac{-3 - 1 + j\sqrt{7} - 1 - j\sqrt{7}}{3} = -\frac{5}{3}$$

The results are shown in Fig. 2

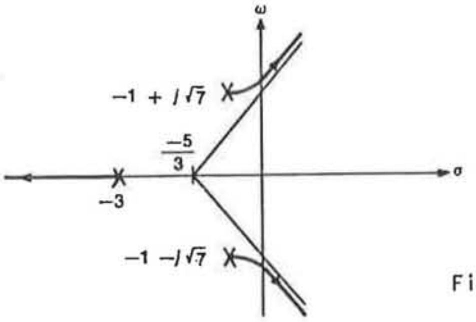


Fig. 2

b) There is a K for which the system becomes unstable. We test the polynomial

$$1 + 2KH_1H_f = 1 + \frac{2K}{(s + 3)(s^2 + 2s + 8)}$$

or
$$P(s) = s^3 + 5s^2 + 14s + 24 + 2K$$

The test function

$$R(s) = \frac{s^3 + 14s}{5s^2 + 24 + 2K} = \frac{s}{5} \frac{(s + 14)}{\left(s^2 + \frac{24 + 2K}{5}\right)}$$

Using the fact that the poles and zeros must alternate on the imaginary axis for a reactance function, we have

$$14 > \frac{24 + 2K}{7} \quad \text{or}$$

$$K < 37$$

as a condition for stability.

c) The overall transfer function is

$$H(s) = \frac{2K/(s + 3)(s^2+2s + 8)}{1 + 2K/(s + 3)(s^2 + 2s + 8)}$$

$$= \frac{2K}{s^3 + 5s^2 + 14s + 24 + 2K}$$

The system is bibo and output-Lyapunov asymptotically stable if $K < 37$. If $K = 37$ there are simple poles on the $j\omega$ axis and the system is output-Lyapunov but not bibo or asymptotically stable.

• PROBLEM 13-107

Fig. 1 shows a block diagram of a simplified minesweeper, K_m is a variable non-negative constant.

(a) Plot a root locus for the system; label important points.
(b) Can the system ever be stable?
(c) Give an analogue computer realization for the complete (compensated minesweeper) system; how many state variables are there? Discuss implementation of your analogue realization where K_m is variable.

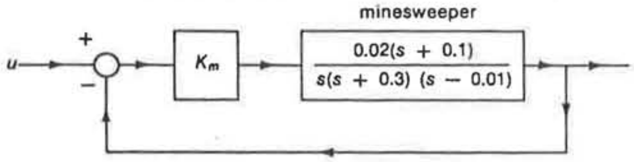


Fig. 1

Solution: We have,

$$K = 0.02K_m \text{ and } H_1 H_f = \frac{s + 0.1}{s(s + 0.3)(s - 0.01)}.$$

The asymptotes are at angles

$$\frac{(2k + 1)\pi}{1 - 3} = \frac{-\pi}{2}, \frac{-3\pi}{2}, \ldots \text{ for } k = 0, 1, \ldots$$

and intersect at the real axis point

$$\sigma_0 = \frac{[0 + (-0.3) + 0.01] - (-0.1)}{3 - 1} = \frac{-0.19}{2} = -0.095.$$

We then have for (a) the root locus of Fig. 2

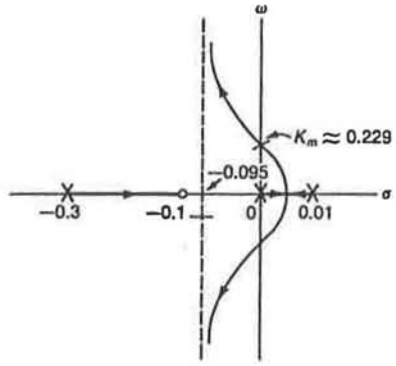


Fig. 2

(b) From the curve, it is clear that there are $K_m > 0$ for which the system can be stable. The "characteristic polynomial" is

$$P(s) = s(s + 0.3)(s - 0.01) + K(s + 0.1)$$

$$= s^3 + 0.29s^2 + (K - 0.003)s + 0.1K,$$

from which the "stable" K can be determined by the Hurwitz test applied to

$$R(s) = \frac{s^3 + (K - 0.003)s}{0.29s^2 + 0.1K}$$

$$\begin{array}{r|l}
 & \frac{1}{0.29}s \\
0.29s^2 + 0.1K & s^3 + (K - 0.003)s \\
 & s^3 + \frac{10}{29}Ks \\
\hline
 & \left(\frac{19}{29}K - 0.003\right)s
\end{array}$$

$$\begin{array}{r|l}
 & \frac{0.29}{\left(\frac{19}{29}K - 0.003\right)}s \\
\left(\frac{19}{29}K - 0.003\right)s & 0.29s^2 + 0.1K \\
 & 0.29s^2 \\
\hline
 & 0.1K
\end{array}$$

$$\begin{array}{r|l}
 & \left(\frac{19}{29}K-0.003\right)s/0.1K \\
0.1K & \left(\frac{19}{29}K-0.003\right)s
\end{array}$$

The factors have to be positive, therefore

$$\frac{19}{29}K - 0.003 > 0 \quad \text{where} \quad K = 0.02K_m$$

is required for stability, or

$$K_m > \frac{29 \times 3}{19 \times 20} = \frac{87}{380} \approx 0.229.$$

(c) We desire the transfer function

$$H(s) = \frac{K_m \frac{0.02(s+0.1)}{s(s+0.3)(s-0.01)}}{1+K_m \frac{(0.02)(s+0.1)}{s(s+0.3)(s-0.01)}} = \frac{Ks+0.1K}{s^3+0.29s^2+(K-0.003)s+0.1K}$$

Setting

$$H(s) = \frac{Y(s)}{U(s)}$$

we obtain

$$\frac{Y(s)}{U(s)} = \frac{Ks + 0.1K}{s^3 + 0.29s^2 + (K-0.003)s + 0.1K}$$

cross multiplying and dividing by s^3

$$\left[1 + \frac{0.29}{s} + \frac{K-0.003}{s^2} + \frac{0.1K}{s^3}\right]Y(s) = \left[\frac{K}{s^2} + \frac{0.1K}{s^3}\right]U(s)$$

$$Y(s) = \left[\frac{-0.29}{s} - \frac{K-0.003}{s^2} - \frac{0.1K}{s^3}\right]Y(s) + \left[\frac{K}{s^2} + \frac{0.1K}{s^3}\right]U(s)$$

The above formula can be arranged

$$Y(s) = \frac{1}{s}\left\{- 0.29Y + \frac{1}{s}\ [KU - (K-0.003)Y + \frac{1}{s}\ (0.1KU - 0.1KY)]\right\},$$

which gives the analogue computer realization as shown in Fig. 3.

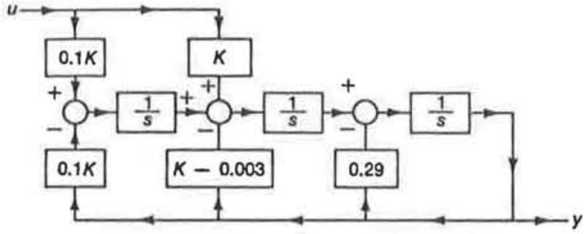


Fig. 3

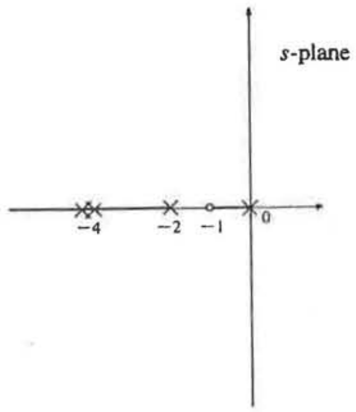
Note that there are three state variables and that if K_m varies, then so must four of the gain blocks, making this realization somewhat unwieldy as far as variation in K_m=K/0.02 is concerned.

• PROBLEM 13-108

The characteristic equation of a feedback control system is

$$1 + F(s) = 1 + \frac{K(s+1)}{s(s+2)(s+4)^2}$$

Determine the effect of the gain K from the root locus.



Solution: First we plot the poles and zeros of the characteristic equation.

0,-2 are poles of the first order; -4 is a pole of the second order.

The root loci on the real axis are shown as heavy lines; they must be located to the left of an odd number of poles and zeros.

From

$$\sigma_A = \frac{1}{n_p - n_z} (\Sigma p_i - \Sigma z_j)$$

we find the intersection of the asymptotes

$$\sigma_A = \frac{-2 + 2(-4) - (-1)}{4 - 1} = -3$$

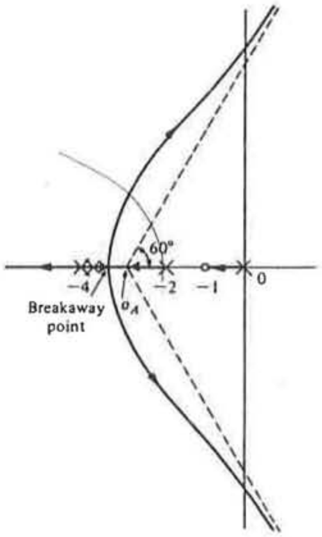
and the angles of the asymptotes are

$$\phi_A = +60° \qquad q = 0$$

$$\phi_A = +180° \qquad q = 1$$

$$\phi_A = 300° \qquad q = 2$$

Since $n_p - n_z = 3$, we have three asymptotes. The root loci must begin at the poles and therefore two loci leave the second order pole at s = -4. Having the asymptotes and the breakaway point we draw the root locus.



From the root locus we conclude that if K is sufficiently increased the system becomes unstable.

• PROBLEM 13-109

We desire to plot the root locus for the characteristic equation of a system when

$$1 + \frac{K}{s(s + 4)(s + 4 + j4)(s + 4 - j4)} = 0 \qquad (1)$$

as K varies from zero to infinity.

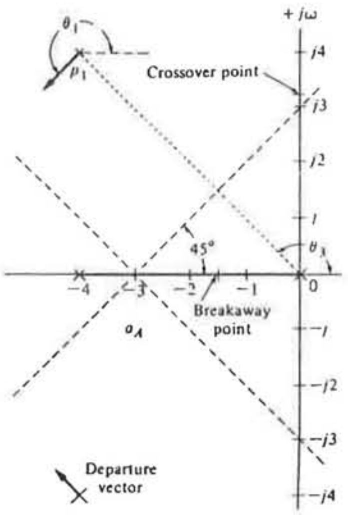


Fig. 1

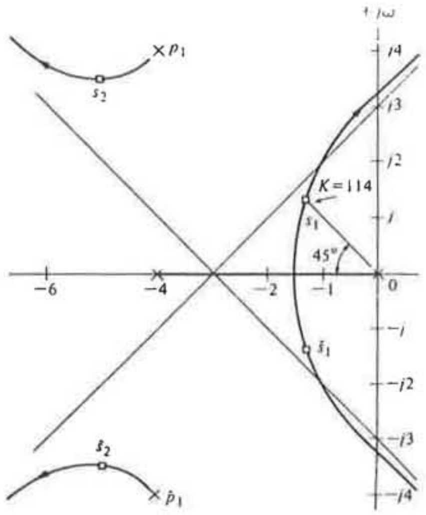


Fig. 2

Solution: The poles are located on the s-plane as shown in Fig. 1. Let s be a point on the complex plane. Let θ_{p_i} be the angle that pole p_i makes with s, and let θ_{z_j} be the angle that zero z_j makes with s. Then, if s is a point on the root locus this relationship must hold, for $K > 0$:

$$\sum_i \theta_{p_i} - \sum_j \theta_{z_j} = (2n + 1)180°) \tag{2}$$

On the real axis, this condition can be met only if the total number of poles and zeros to the right is odd. (Poles and zeros on the left contribute 0°, while those on the right contribute ±180°.) In our case, this condition exists only between s = 0 and s = -4. Thus, a segment of the root locus exists on the real axis between s = 0 and s = -4. Since the number of poles n_p is equal to four and there are no zeros, we have $n_p - n_z = 4$ separate loci. The angles of the asymptotes are

$$\phi_A = \frac{(2q + 1)}{n_p - n_z}\, 180° \;, \quad q = 0,1,2,3, \tag{3}$$

$$\phi_A = +45°,\ 135°,\ 225°,\ 315°$$

The center of the asymptotes is

$$\sigma_A = \frac{1}{n_p - n_z}\left(\sum_i \mathrm{Re}\{p_i\} - \sum_j \mathrm{Re}\{z_j\}\right) \tag{4}$$

$$\sigma_A = \frac{-4 - 4 - 4}{4} = -3 \; .$$

Then the asymptotes are drawn as shown in Fig. 1.

The breakaway point is estimated by evaluating

$$K = p(s) = -s(s + 4)(s + 4 + j4)(s + 4 - j4) \qquad (5)$$

between s = -4 and s = 0.

Breakaway points exist between two adjacent poles or two adjacent zeros (including zeros at infinity) on the real axis, and sometimes at other points not on the real axis. At a breakaway point, the value of K in terms of s is an extreme point. We expect the breakaway point to lie between s = -3 and s = -1 and, therefore, we search for a maximum value of p(s) in that region. The resulting values of p(s) for several values of s are given in the table. The maximum of p(s) is found to lie at approximately s = -1.5 as indicated in the table. A more accurate estimate of the breakaway point is

p(s)	0	51	68.5	80	85	75	0
s	-4.0	-3.0	-2.5	-2.0	-1.5	-1.0	0

normally not necessary or worthwhile. The breakaway point is then indicated on Fig. 1.

The characteristic equation is rewritten as

$$s(s + 4)(s^2 + 8s + 32) + K = s^4 + 12s^3 + 64s^2 + 128s + K = 0 \quad (6)$$

Therefore, the Routh-Hurwitz array is

$$\begin{array}{c|ccc} s^4 & 1 & 64 & K \\ s^3 & 12 & 128 & \\ s^2 & b_1 & K & \\ s^1 & c_1 & & \\ s^0 & K & & \end{array}$$

where

$$b_1 = \frac{12(64) - 128}{12} = 53.33 \text{ and } c_1 = \frac{53.33(128) - 12K}{53.33} \qquad (7)$$

For a certain value of K, the system becomes marginally stable. At that point, a row in the Routh-Hurwitz array becomes all zero. The auxiliary equation of the row above then gives the points at which the locus crosses the imaginary axis. Here, we can only set $c_1 = 0$.

Hence the limiting value of gain for stability is K = 570 and the roots of the auxiliary equation are

$$53.33s^2 + 570 = 53.33(s^2 + 10.6) = 53.33(s + j3.25)(s - j3.25). \quad (8)$$

The points where the locus crosses the imaginary axis is shown in Fig. 1.

The angle of departure at the complex pole p_1 can be estimated by utilizing the angle criterion as follows:

$$\theta_1 + 90° + 90° + \theta_3 = 180° , \quad (9)$$

where θ_3 is the angle subtended by the vector from pole p_3. The angles from the pole at $s = -4$ and $s = -4 - j4$ are each equal to 90°. Since $\theta_3 = 135°$, we find that

$$\theta_1 = -135° = +225°$$

as shown in Fig. 1.

Utilizing all the information obtained from the steps of the root locus method, the complete root locus is plotted by using a protractor or Spirule to locate points that satisfy the angle criterion. The root locus for this system is shown in Fig. 2. When complex roots near the origin have a damping ratio of $\zeta = 0.707$, the gain K can be determined graphically as shown in Fig. 2. The vector lengths to the root location s_1 from the open-loop poles are evaluated and result in a gain at s_1 of

$$K = |s_1||s_1 + 4||s_1 - p_1||s_1 - \hat{p}_1| \quad (10)$$

$$= (1.3)(3.2)(4.4)(6.2) = 114 .$$

The remaining pair of complex roots occurs at s_2 and $\hat{s}_2$ when $K = 114$. The effect of the complex roots at s_2 and $\hat{s}_2$ on the transient response will be negligible compared to the roots s_1 and $\hat{s}_1$. This fact can be ascertained by considering the damping of the response due to each pair of roots. The damping due to s_1 and $\hat{s}_1$ is

$$e^{-\zeta_1 \omega n_1 t} = e^{-\sigma_1 t} , \quad (11)$$

and the damping factor due to s_2 and $\hat{s}_2$ is

$$e^{-\zeta_2 \omega n_2 t} = e^{-\sigma_2 t} , \quad (12)$$

where σ_2 is approximately five times as large as σ_1 . Therefore, the transient response term due to s_2 will decay much more rapidly than the transient response term due to s_1 . Thus the response to a unit step input may be written as

$$c(t) = 1 + c_1 e^{-\sigma_1 t} \sin(\omega_1 t + \theta_1) + c_2 e^{-\sigma_2 t} \sin(\omega_2 t + \theta_2)$$

$$\cong 1 + c_1 e^{-\sigma_1 t} \sin(\omega_1 t + \theta_1) . \quad (13)$$

The complex conjugate roots near the origin of the s-plane relative to the other roots of the closed-loop system are labeled the dominant roots of the system since they represent or dominate the transient response. The relative dominance of the roots is determined by the ratio of the real parts of the complex roots and will result in reasonable dominance for ratios exceeding five.

Of course, the dominance of the second term of Eq. 13 also depends upon the relative magnitudes of the coefficients c_1 and c_2. These coefficients, which are the residues evaluated at the complex roots, in turn depend upon the location of the zeros in the s-plane. Therefore, the concept of dominant roots is useful for estimating the response of a system but must be used with caution and with a comprehension of the underlying assumptions.

• PROBLEM 13-110

In some control systems a positive feedback inner loop may appear. This loop is usually stabilized by the outer loop. For the system shown in Fig. 1, with the positive feedback, sketch the root locus plot.

Assume that H(s) = 1

$$G(s) = \frac{K(s+3)}{(s+4)(s^2+2s+2)} \qquad K>0$$

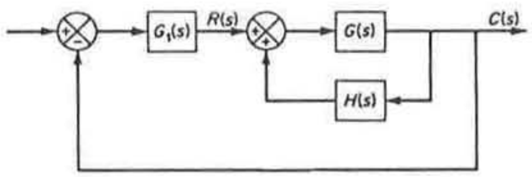


Fig. 1

Solution: The procedure for plotting positive feedback control system is similar to that for the negative feedback control system with slight modifications.

We write the transfer function of the inner loop

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)\,H(s)}$$

and the characteristic equation

$$1 - G(s)\,H(s) = 0$$

or $\quad G(s)\,H(s) = 1.$

This is equivalent to two equations:

$$\angle G(s)\,H(s) = 0° \pm k360° \quad (k=0,\ 1,\ 2,\ \ldots)$$

$$|G(s)\,H(s)| = 1$$

The total sum of all angles from the open-loop poles and zeros is equal to $0° \pm k360°$. We use the following rules for constructing root loci of the positive feedback system.

1. If the total number of real zeros and real poles to the right of a test point on the real axis is even, then this test point lies on the root locus.

2. Angles of asymptotes $= \dfrac{\pm 360° k}{n - m}$

   where

   n = number of finite poles of G(s) H(s)
   m = number of finite zeros of G(s) H(s)

3. To calculate the angle of departure (or angle of arrival) from a complex open-loop pole (or at a complex zero) we subtract from 0° the sum of all the angles of the complex quantities from all the other poles and zeros to the complex pole (or complex zero) in question, including the appropriate signs.

For the positive feedback system the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)\,H(s)} = \frac{k(s + 3)}{(s + 4)(s^2 + 2s + 2) - k(s + 3)}$$

1. We find the open-loop poles

   $s = -1 + j$, $s = -1 - j$, $s = -4$ and zero $s = -3$.
   We note that when k increases from 0 to ∞, the closed-loop poles start at the open-loop poles and terminate at the open-loop zeros.

2. The root loci on the real axis exist between -3 and $+\infty$ and between -4 and $-\infty$.

3. The asymptotes of the root loci.
   We have
   Angle of asymptote $= \dfrac{\pm k360°}{3-1} = \pm 180°$
   That means that root loci branches are located on the real axis.

4. Breakaway points and break-in points.
   The characteristic equation is

   $$(s + 4)(s^2 + 2s + 2) - k(s + 3) = 0$$

   we calculate k and $\dfrac{dk}{ds}$

   $$k = \frac{(s + 4)\ (s^2 + 2s + 2)}{(s + 3)}$$

   $$\frac{dk}{ds} = \frac{2s^3 + 15s^2 + 36s + 22}{(s + 3)^2}$$

   Solving the equation we obtain

   $$2s^3 + 15s^2 + 36s + 22 = 2(s + 0.9)(s^2 + 6.6s + 12.2) =$$

   $$2(s + 0.9)(s + 3.3 - j1.15)(s + 3.3 + j1.15)$$

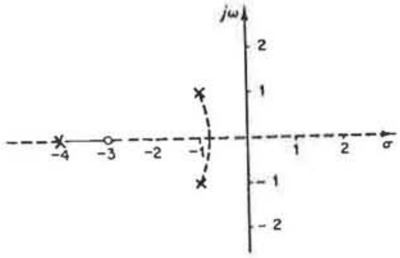
points s = -3.3 + j1.15 and s = -3.3 - j1.15 do not satisfy the angle condition. At s = -0.9 the value of k is positive. Thus the break-in point is s = -0.9.

5. The angle of departure of the root locus from a complex pole. For the complex pole s = -1 + j, the angle of departure ϕ is

$$45° - 29° - 90° - \phi = 0°$$
$$\phi = -74°$$

and for s = - 1 - j, $\phi = 74°$.

6. The test point should be in the broad neighborhood of the imaginary axis and the origin; we apply the angle condition.



Root-locus plot for the positive feedback system with
$G(s) = \dfrac{K(s+3)}{(s+4)(s^2+2s+2)} \quad K > 0$
$H(s) = 1$

For the positive feedback system we obtain the following root-locus plot using results 1 - 6.

NYQUIST-BODE

• PROBLEM 13-111

The closed-loop system is shown in Fig. (1).

Using the Nyquist criterion determine the critical value of k for stability of the system.

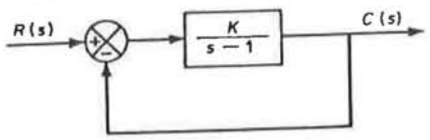


Fig. 1

Solution: The gain function of the system is

$$G(s) = \frac{k}{s-1}$$

Let us draw the polar plot of G(jw)

$$G(jw) = \frac{k}{jw - 1} = \frac{-k - jwk}{w^2 + 1}$$

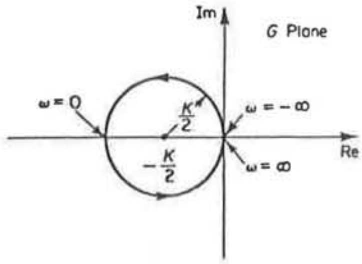


Fig. 2

The direction is counterclockwise. There is one pole of G(s) in the right-half s plane, thus P = 1. The condition for stability of a closed-loop system is Z = 0. Therefore N must be equal to −1, since N = Z − P or there must be one counterclockwise encirclement of the −1 + j0 point for stability.

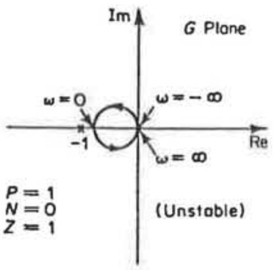


Fig. 3

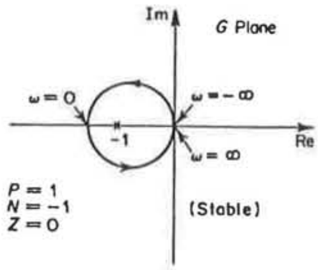


Fig. 4

For stability of the system we have $k>1$

$k = 1$ is the stability limit.

The two figures 3, 4, illustrate the results.

• PROBLEM 13-112

For the system whose open-loop transfer function is

$$\frac{\Theta_{on}}{\Theta} = \frac{10}{D(1 + \frac{D}{4})(1 + \frac{D}{16})}$$

determine the stability.

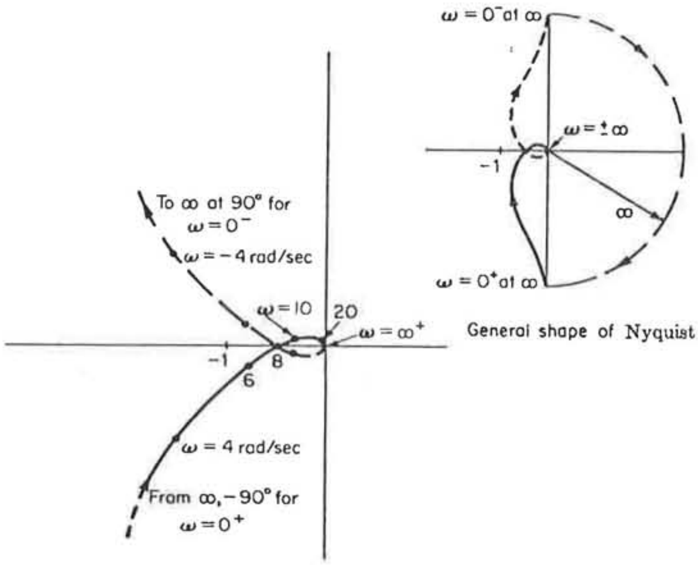
Solution: Using the Nyquist Criterion we have:

$$D = jw$$

then

$$\frac{\Theta_{on}}{\Theta} = \frac{10}{jw(1 + \frac{jw}{4})(1 + \frac{jw}{16})} = \frac{640}{jw(jw + 4)(jw + 16)}$$

$$= \frac{640}{-20w^2 + j(64w - w^2)}$$



General shape of Nyquist plot

Accurate plot in the region of (−1, 0)

We can draw the Nyquist plot as shown in the figure.

We see that the point (-1,0) is not encircled and the system is stable.

• PROBLEM 13-113

Determine for what values of k the system with the following open-loop transfer function is stable

$$G(s)\ H(s) = \frac{k}{s(s + 1)(2s + 1)}$$

Solution: Let s = jw we have

$$G(jw)\ H(jw) = \frac{k}{jw(jw + 1)(2jw + 1)} =$$

$$\frac{k}{-3w^2 + jw(1 - 2w^2)}$$

We see that the open-loop transfer function has no poles in the right half s-plane. For the system to be stable is thus enough that the Nyquist plot does not encircle the -1 + 0j point. To find the point where the Nyquist plot crosses the negative real axis let I_m G(jw) H(jw) = 0 thus

$$1 - 2w^2 = 0$$

and $w = \pm \frac{1}{\sqrt{2}}$

$$G\left(j\frac{1}{\sqrt{2}}\right) H\left(j\frac{1}{\sqrt{2}}\right) = \frac{-2k}{3}$$

To get the critical value of k let

$$-\frac{2k}{3} = -1$$

$$k = \frac{3}{2}$$

We have

$$0<k<\frac{3}{2}$$

for stability of the system.

• PROBLEM 13-114

A closed-loop system has the following open-loop transfer function

$$G(s)\ H(s) = \frac{k(s+3)}{s(s-1)}$$

Investigate the stability of the system.

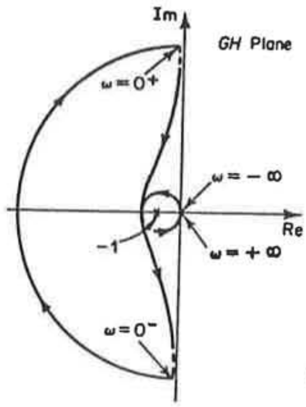


Fig. 1

Polar plot of the system

Solution: It is clear that the open-loop transfer function has one pole for s = 1 in the right-half s plane, therefore the open-loop system is unstable. From the Nyquist plot see Fig. (1), we see that the point - 1 + j0 is encircled by the G(s) H(s) locus once in the counterclockwise direction. Thus, N = -1. Since P = 1 from Z = N + P we find Z to be zero, that indicates that there is no zero of 1 + G(s) H(s) in the right-half s plane and the closed-loop system is stable.

It is worthwhile to note that in the above example an unstable open-loop system becomes stable when the loop is closed.

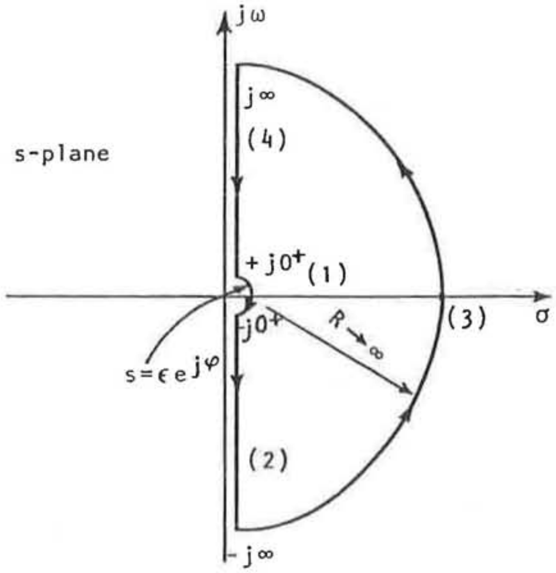
The loop transfer function of a single-loop feedback control system is

$$G(s)\ H(s) = \frac{k}{s(s + \alpha)}$$

where k and α are positive constants. Investigate the stability of the system.

Solution: We start with construction of the Nyquist path for the system. Since $\alpha>0$, G(s) H(s) does not have any poles in the right-half s-plane, thus $P_0 = P_{-1} = 0$.

G(s) H(s) has a pole at the origin, the Nyquist path must not pass through any singularity of G(s) H(s) thus we draw a small semicircle around s = 0.



THE NYQUIST PATH FOR $G(s)H(s)=\frac{K}{s(s+\alpha)}$

We divided the Nyquist path into four sections.

The points of section (1) may be represented by the phasor

$$s = \varepsilon e^{j\phi}$$

where $\varepsilon \to 0$ and ε, ϕ denote the magnitude and phase.

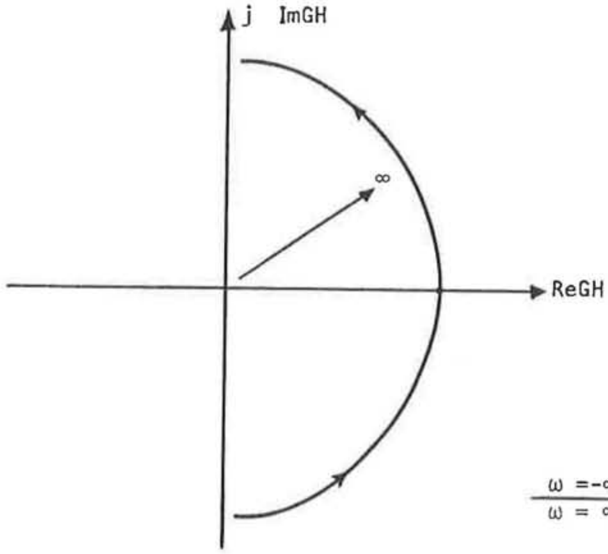
We see that as the Nyquist path is traversed from $+j0^+$ to $-j0^+$ along section (1) ϕ changes from $+90°$ to $-90°$.

Since $e^{j\phi} = \cos\phi + j \sin\phi$

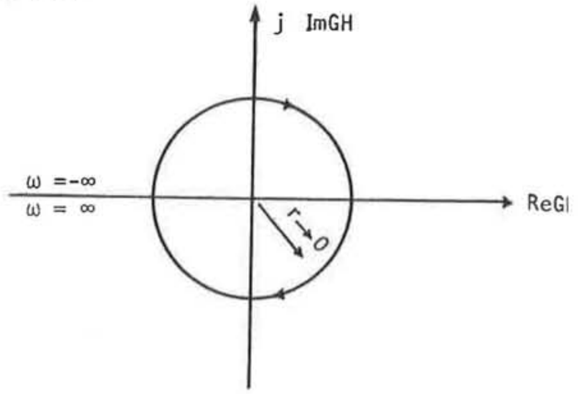
for $\phi = +90°$ $\quad s = j\varepsilon$.

The Nyquist plot for section (1) is

$$G(s)H(s) = \frac{K}{s(s+\alpha)}\Bigg|_{s=\varepsilon e^{j\phi}} = \frac{K}{\varepsilon e^{j\phi}(\varepsilon e^{j\phi}+\alpha)} \xrightarrow[\varepsilon\to 0]{} \frac{K}{\alpha\varepsilon e^{j\phi}} = \infty e^{-j\phi}$$



THE NYQUIST PLOT OF SECTION (1)



THE NYQUIST PLOT OF SECTION (3)

The points corresponding to section (1) have an infinite magnitude and the phase is opposite to that of the s-plane locus.

For section (3) we have

$$s = Re^{j\phi}$$

where $R\to\infty$, and ϕ changes from -90° to +90°.

$$G(s)H(s) = \frac{k}{s(s+\alpha)}\Bigg|_{s=Re^{j\phi}} = \frac{k}{Re^{j\phi}(Re^{j\phi}+\alpha)} \xrightarrow[R\to\infty]{} \frac{k}{R^2e^{2j\phi}} = 0e^{-2j\phi}$$

We see that the magnitude is infinitesimally small, and the phasor rotates 2 x 180° = 360° in the clockwise direction.

We are left with sections (2) and (4). For section (4) we substitute s = jw

$$G(s)H(s) = G(jw)H(jw) = \frac{k}{jw(jw+\alpha)} = \frac{k(-w^2 - j\alpha w)}{w^4 + \alpha^2 w^2}$$

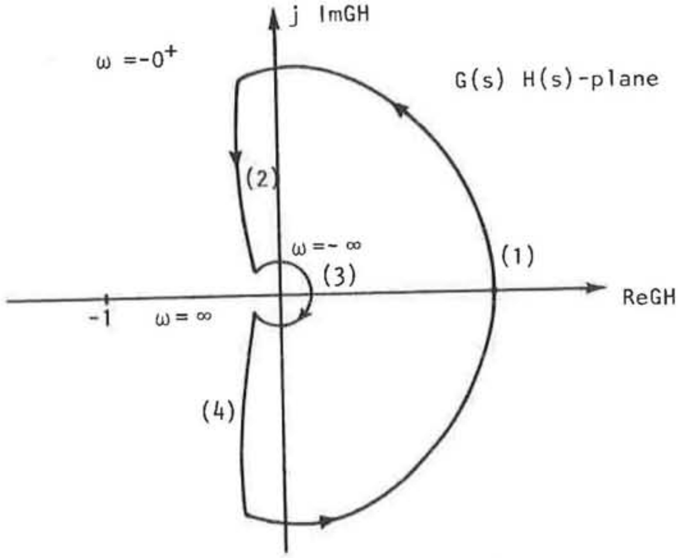
To find the intersect of G(jw)H(jw) on the real axis we equate

$$\text{Im}G(jw)H(jw) = 0$$

$$-\frac{k\alpha w}{w^4 + w^2\alpha^2} = \frac{-k\alpha}{w(w^2+\alpha^2)} = 0$$

which gives $w = \infty$.

We can draw the complete Nyquist plot of G(s)H(s).



THE NYQUIST PLOT OF G(s) H(s)

We see that

$N_0 = N_{-1} = 0$ where N_0 is the number of encirclements of the origin made by G(s)H(s).

Since $G(s)H(s) = \frac{k}{s(s+\alpha)}$, $k>0$, $\alpha>0$

we have $Z_0 = 0$ and $P_0 = 0$.

Since $P_{-1} = P_0 = 0$

we have $Z_{-1} = N_{-1} + P_{-1} = 0$

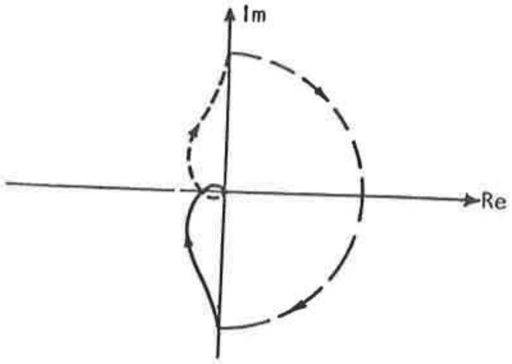
We conclude that the closed-loop system is stable.

• PROBLEM 13-116

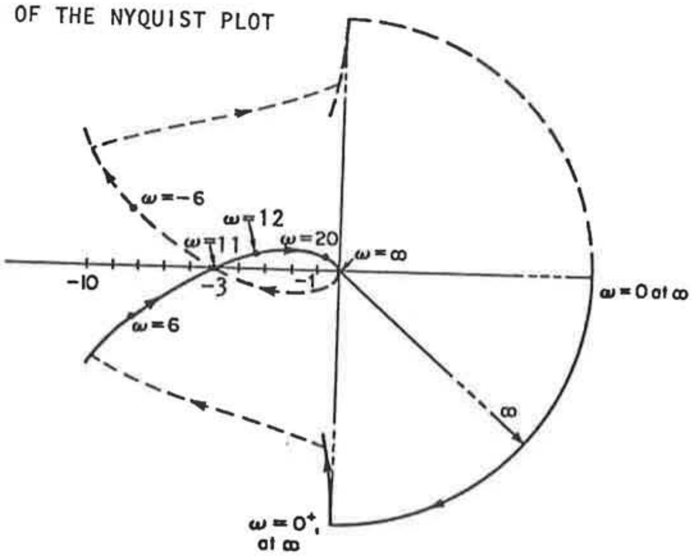
The open loop transfer function of a system is

$$\frac{\Theta_{on}}{\Theta} = \frac{80}{D(1+\frac{D}{5})(1+\frac{D}{25})}$$

Determine the stability of the system.



THE GENERAL SHAPE OF THE NYQUIST PLOT



ACCURATE PLOT

Solution: Let us substitute jw = D. We have

$$\left(\frac{\Theta_{on}}{\Theta}\right)_{D=jw} = \frac{80}{jw(1+\frac{jw}{5})(1+\frac{jw}{25})} = \frac{80}{-\frac{6}{25}w^2 + j(w - \frac{w^3}{125})}$$

For amplitude and phase we get:

$$G_0 = \frac{80}{w(1+\frac{w^2}{25})^{\frac{1}{2}}(1+\frac{w^2}{625})^{\frac{1}{2}}}$$

$$\phi_0 = -90° - \tan^{-1}\frac{w}{5} - \tan^{-1}\frac{w}{25}$$

For some values of w we shall calculate G_0 and ϕ_0.

w =	0+	1	2	4	6	10	20	50	∞
G_0 =	∞	78	37	15	8	3.32	0.75	0.07	0
ϕ_0 =	−90°	−103°	−116°	−137°	−153°	−175°	−204°	−237°	−270°

For the negative values of w, we plot the mirror images of the positive values. The point $(-1, 0_j)$ is encircled, hence the system with the open loop transfer function

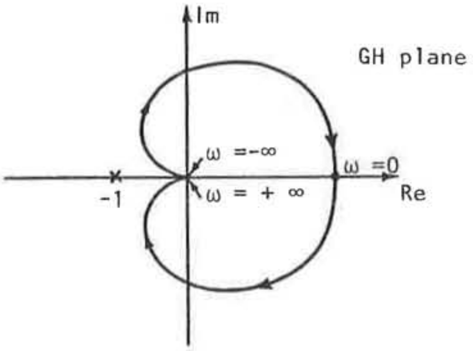
$$\frac{80}{D(1 + \frac{D}{5})(1 + \frac{D}{25})} \quad \text{is unstable.}$$

• PROBLEM 13-117

The open-loop transfer function of a closed-loop system is given by

$$G(s)H(s) = \frac{k}{(T_1 s + 1)(T_2 s + 1)}$$

Determine the stability of the system.



POLAR PLOT OF $G(j\omega)\ H(j\omega) = \frac{K}{(T_1 j\omega + 1)\ (T_2 j\omega + 1)}$

Solution: We shall plot $G(jw)H(jw)$.

Substituting $s = jw$ one gets

$$G(jw)H(jw) = \frac{k}{(T_1 jw + 1)(T_2 jw + 1)}$$

We transform the above equation to get the real and imaginary parts

$$G(jw)H(jw) = \frac{k(1 - T_1 jw)(1 - T_2 jw)}{(T_1 jw + 1)(T_1 jw - 1)(T_2 jw + 1)(T_2 jw - 1)}$$

$$= \frac{k(1 - T_1 jw)(1 - T_2 jw)}{(T_1^2 w^2 + 1)(T_2^2 w^2 + 1)} = \frac{k(1 - T_1 T_2 w^2) - jw(T_1 + T_2)k}{(T_1^2 w^2 + 1)(T_2^2 w^2 + 1)}$$

The angle is given by

$$\tan^{-1}\left[\frac{\text{imaginary part}}{\text{real part}}\right] = \tan^{-1}\left[\frac{-w(T_1 + T_2)k}{k(1 - T_1 T_2 w^2)}\right]$$

Then

$w \to 0+ \Rightarrow +1 - j0$ angle $\approx -\frac{1}{w}$

so $\angle -90°$

$w \to \infty \Rightarrow -0 - j0$ angle $\approx -\frac{1}{w^2}$

so $\angle -180°$

$w \to -\infty \Rightarrow -0 + j0$ angle $\approx \frac{1}{w^2}$

$\angle 180°$

$w \to 0^- \Rightarrow +1 + j0$ angle $\approx \frac{1}{w}$

$\angle 90°$

We have all the information to plot $G(jw)H(jw)$.

Since $G(s)H(s)$ does not have any poles in the right-half s plane and the point $(-1, 0j)$ is not encircled by the $G(jw)H(jw)$ locus, the system is stable for

$K > 0$
$T_1 > 0$
$T_2 > 0$.

• PROBLEM 13-118

The open-loop transfer function of the system is

$$G(s)H(s) = \frac{k}{S(T_1 s + 1)(T_2 s + 1)}$$

Investigate the stability of the system in two cases

a) the value of the gain k is small
b) the value of the gain k is large.

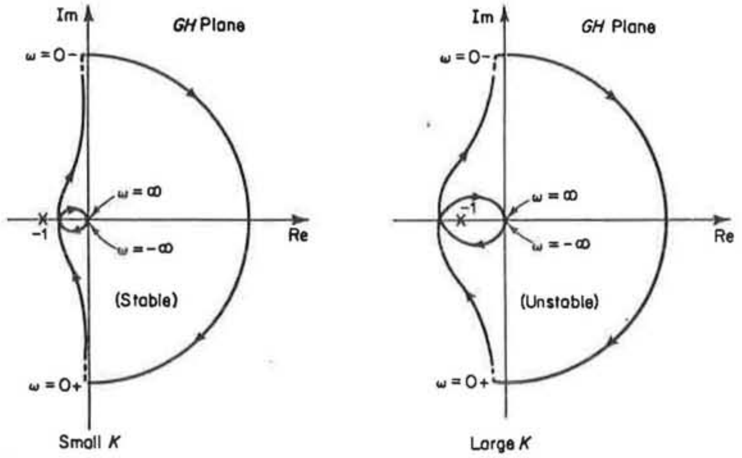
Solution:

We shall draw the Nyquist plot for the small and large values of k.

The number of poles of $G(s)H(s)$ in the right-half s plane is zero. Thus for stability of the system it is necessary that

$$Z = N = 0$$

or that the $G(s)H(s)$ plot does not encircle $(-1, 0j)$. From the plot we conclude that for small values of K there is no encirclement of the $(-1, j0)$ point and the system is stable.



For small k we have

$$P = 0, \ N = 0, \ Z = 0.$$

For large K there are two encirclements of the point (-1, 0j) in the clockwise direction indicating two closed-loop poles in the right-half s plane and the system is unstable.

For large K we have

$$P = 0, \ N = Z = 2.$$

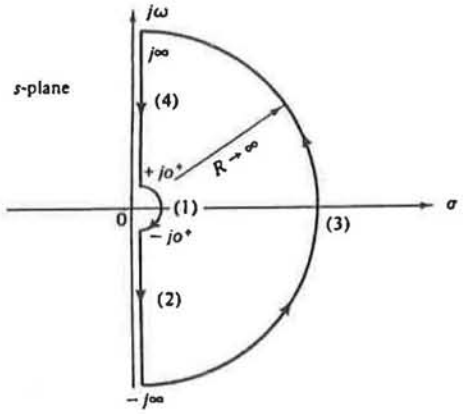
In the case of the above system, large K increases accuracy but decreases stability.

• PROBLEM 13-119

The loop transfer function of a control system with a single feedback loop is

$$G(s)H(s) = \frac{K(s - 1)}{s(s + 1)}$$

Using the Nyquist criterion determine for which values of K the system is stable.



Solution:

We shall start with drawing the Nyquist path of the system.

To obtain the Nyquist plot let us investigate the separate sections of the path.

Section (1).

Let $s = \varepsilon e^{j\phi}$ where $\varepsilon \to 0$

we have

$$G(s)H(s)\Big|_{s=\varepsilon e^{j\phi}} = \frac{k}{s} = \infty e^{-j(\phi + \pi)}$$

For this section the magnitude is infinite and the angle changes from $+90^\circ$ to -90° counterclockwise.

Section (3).

$$s = R e^{j\phi}$$

$$\lim_{s\to\infty} G(s)H(s) = \lim_{s\to\infty} \frac{k}{s} = 0\, e^{-j\phi}$$

The magnitude is zero and the angle changes 180° in the clockwise direction.

Section (4).

Let $s = jw$

$$G(jw)H(jw) = \frac{K(jw - 1)}{jw(jw + 1)} = K\,\frac{2w + j(1 - w^2)}{w\,(w^2 + 1)}$$

From the equation

$$\text{Im } G(jw)H(jw) = 0$$

We obtain

$$W = \pm 1 \text{ rad/sec.}$$

Then $G(j1)H(j1) = K$.

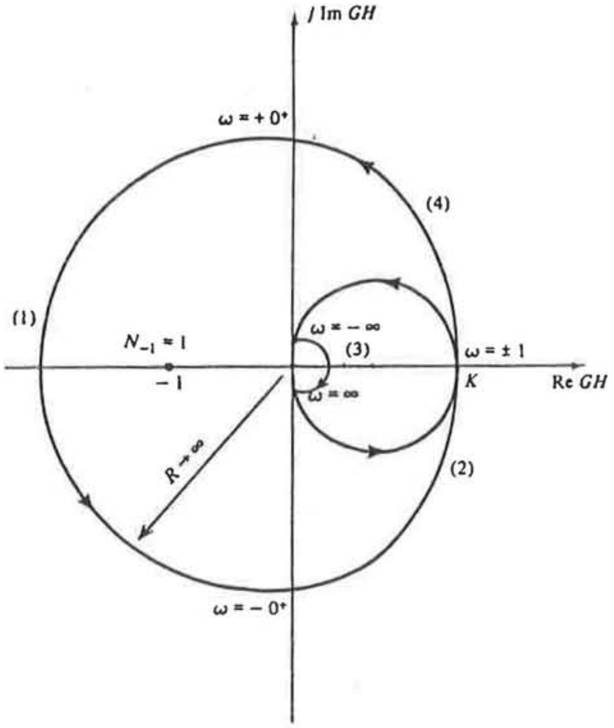
Gathering the results we can draw the Nyquist plot of

$$G(s)H(s) = \frac{K(s - 1)}{s(s + 1)}$$

We see that

$$Z_0 = 1, \quad P_0 = P_{-1} = 0$$

$$N_0 = N_{-1} = 1$$



THE NYQUIST PLOT

then $Z_{-1} = N_{-1} + P_{-1} = 1$

Thus the closed-loop system is unstable. From the Nyquist plot we conclude that the system can not be stabilized by changing the value of parameter K.

• PROBLEM 13-120

Below is shown the Nyquist diagram for a system whose open-loop transfer function may be one of the following

a) no poles and zeros in the right-half s plane

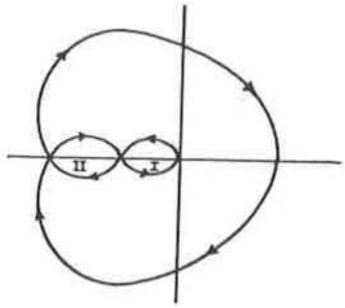
b) no poles and one zero in the right-half s plane

c) one pole and no zeros in the right-half s plane

d) two poles and no zeros in the right-half s plane

e) two poles and two zeros in the right-half s plane.

Decide on stability for each case if the point $-1 + j0$ is located first in region I and then in II. Assume that the feedback loop is closed on each one of the above transfer functions.



Solution:

First of all let us note that for all the transfer functions point in region I is not encircled and a point in region II : encircled twice. The general formula is

$$N_{-1} = P_{-1} - Z_{-1}$$

where P_{-1} is the number of poles of 1 + G, and $P_{-1} = P_0$.

For stability the number Z_{-1} of zeros of 1 + G must be zero. Thus, for stability $N_{-1} = P_{-1}$ or P_0

a) $P_0 = 0$ so N_{-1} must be 0, therefore I - stable, II unstable

b) $P_0 = 0$, $N_{-1} = 0$ so I - stable, II - unstable

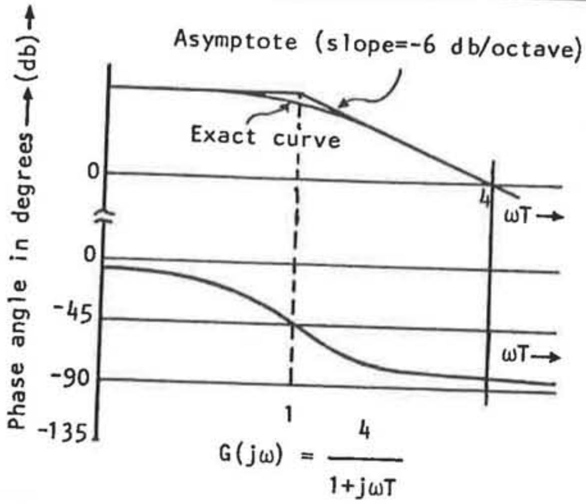
c) $P_0 = 1$ $N_{-1} = 1$ thus I and II unstable

d) $P_0 = 2$ $N_{-1} = 2$ I unstable II stable

e) $P_0 = 2$ $N_{-1} = 2$ I unstable II stable

• PROBLEM 13-121

For the function shown in Fig. (1) determine the stability margin.



$G(j\omega) = \dfrac{4}{1+j\omega T}$

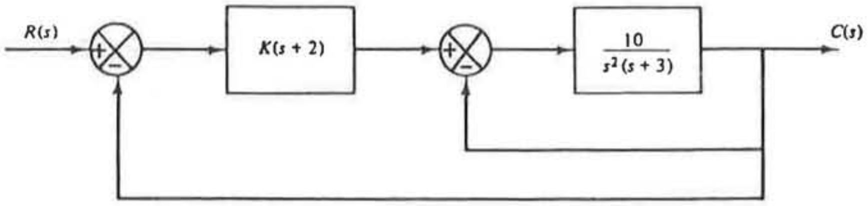
Fig. 1

Solution:

At wT = 4 the magnitude plot crosses the OdB line. On the phase curve the point corresponding to wT = 4 is (=$\tan^{-1}4$) about 75°. The phase margin is then 180° - 75° = 105° and there is no gain margin since the phase never reaches 180°.

• PROBLEM 13-122

Determine the range of parameter K for which the system shown on the block diagram is stable.



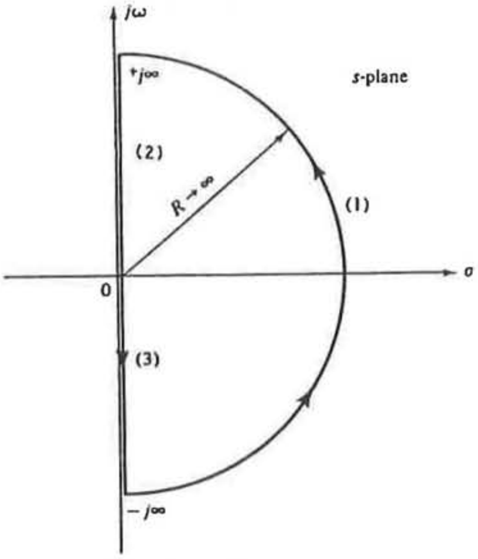
Solution:

The transfer function $G(s) = \frac{10K(s+2)}{s^2(s+3)}$

and the open-loop transfer function is

$$K(s+2)\,\frac{10/s^2(s+3)}{1+10/s^2(s+3)}$$

Since this function does not have any zeros or poles on $j\omega$ axis the Nyquist path is



THE NYQUIST PATH

We shall construct the Nyquist plot for the (1), (2) and (3) sections.

Section (1):

Let $s = R\,e^{j\theta}$ where $R \to \infty$

$$\lim_{s\to\infty} G(s) = \frac{10K}{s^2} = 0e^{-j2\theta}$$

We see that the magnitude is zero and the angle changes from +180° to -180° clockwise, since θ changes from -90° to +90° counterclockwise.

Section (2): $s = j\omega$

$$G(j\omega) = \frac{10K(j\omega+2)}{(10-3\omega^2) - j\omega^3}$$

To rationalize the above fraction we multiply by

$$(10-3\omega^2) + j\omega^2 \ ,$$

thus

$$G(j\omega) = \frac{10K[2(10-3\omega^2) - \omega^4 + j\omega(10-3\omega^2) + j2\omega^3]}{(10-3\omega^2)^2 + \omega^6}$$

From $\quad \text{Im}\{G(j\omega)\} = 0 \quad$ we get

$$\omega = 0 \qquad \text{or} \qquad \omega = \pm\sqrt{10}$$

which are the values of the intersects of the real axis of the G(s) plane.

To determine the intersection of the G(s) plot on the imaginary axis we set

$$\text{Re}\{G(s)\} = 0$$

and obtain

$$\omega^4 + 6\omega^2 - 20 = 0$$

$$\omega = \pm\sqrt{2}$$

Thus the intersects of the real axis are

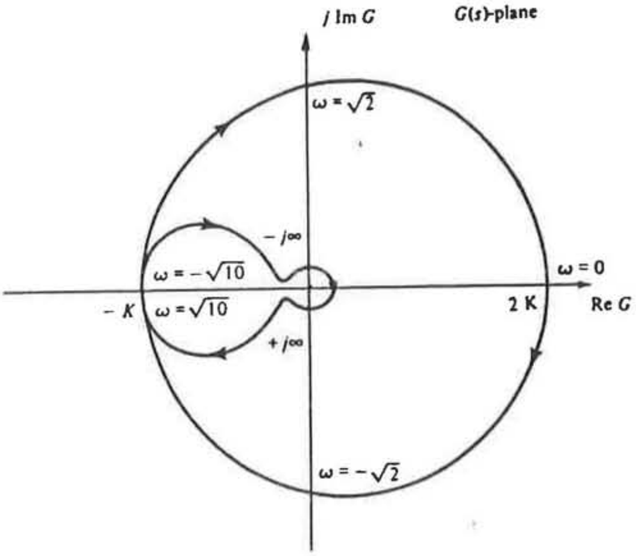
$$G(j0) = 2K$$

$$G(j\sqrt{10}) = -K$$

and the imaginary axis

$$G(j\sqrt{2}) = j10\sqrt{2}\,\frac{K}{3}$$

We have all the information to draw the Nyquist plot.



From the plot we see that $N_0 = -2$ and since $Z_0 = 0$ we get $P_0 = 2$. Thus $P_{-1} = 2$.

From the Nyquist criterion we have

$$N_{-1} = Z_{-1} - P_{-1} = Z_{-1} - 2$$

For the closed-loop system to be stable $Z_{-1} = 0$ and we get $N_{-1} = -2$.

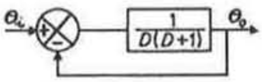
Thus the point $(-1, j0)$ must be encircled twice in the clockwise direction.

We have this when point $(-1, j0)$ is inside the circle, thus the condition for stability is

$$K > 1.$$

• PROBLEM 13-123

The block diagram of a system is



Find the stability condition of the system.

Solution: The open loop transfer function of the system is

$$\frac{\theta_0}{\theta} = \frac{1}{D(D+1)}$$

Let us substitute $D = j\omega$

$$\left(\frac{\theta_0}{\theta}\right)_{D=j\omega} = \frac{1}{j\omega(j\omega+1)} = \frac{1}{j\omega-\omega^2}$$

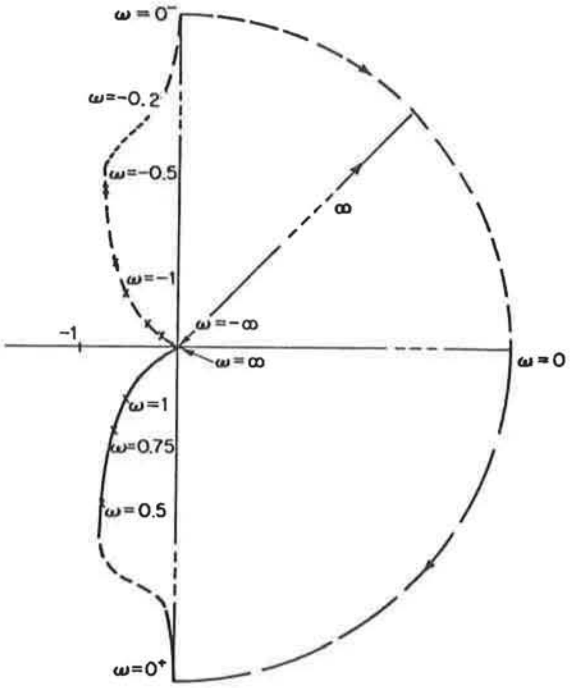
We shall compute the magnitude ratio and phase of the o
loop transfer function

$$G_0 = \frac{1}{\omega(1+\omega^2)^{\frac{1}{2}}}$$

and

$$\phi_0 = -\tan^{-1}\frac{\omega}{-\omega^2} = -\tan^{-1}\frac{1}{-\omega}$$

$$= -90 - \tan^{-1}\omega \ .$$



THE NYQUIST DIAGRAM FOR THE SYSTEM WITH OPEN LOOP TRANSFER FUNCTION 1/[D(D + 1)].

The plot for negative ω is a mirror image of the positiv
plot. Point (-1,0) is not encircled, thus the closed lo
system is stable.

CHAPTER 14

PHASE PLANE ANALYSIS

INITIAL CONDITIONS

• PROBLEM 14-1

Consider the first-order systems described by

$$\dot{x} = -x \qquad (1)$$

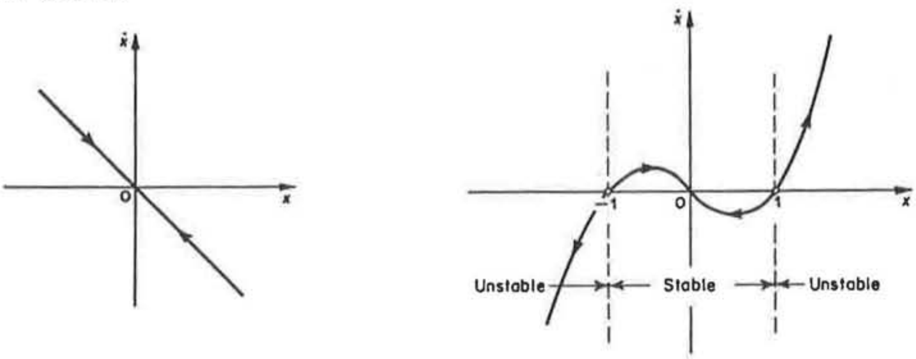
and

$$\dot{x} = -x + x^3 \qquad (2)$$

Draw the phase trajectories and show where the systems are stable and unstable.

Solution: In the phase plane, or $x - \dot{x}$ plane, the phase-plane plot of eq. (1) is a straight line. For any initial condition, $x(0)$, the system returns to its singular point, the origin, after an infinite time.

The starting point of the trajectory is determined by the initial condition $x(0)$. For the system $\dot{x} = -x + x^3$, the trajectory is shown below.



The trajectory is divided into three parts, two unstable and one stable part.

If $x(0) > 1$, then $x(\infty) \to \infty$,