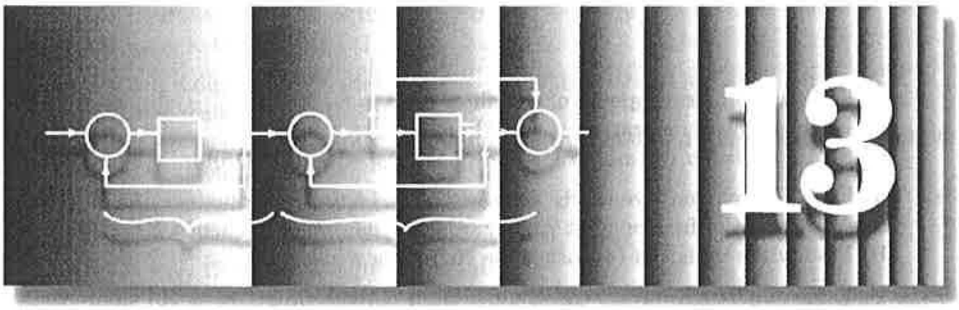


State-Space Analysis

So far we have been describing systems in terms of equations relating certain output to an input (the input-output relationship). This type of description is an *external description* of a system (system viewed from the input and output terminals). As noted in Chapter 1, such a description may be inadequate in some cases, and we need a systematic way of finding system's *internal description*. State space analysis of systems meets this need. In this method, we first select a set of key variables, called the **state variables**, in the system. The state variables have the property that every possible signal or variable in the system at any instant t can be expressed in terms of the state variables and the input(s) at that instant t. If we know all the state variables as a function of t, we can determine every possible signal or variable in the system at any instant with a relatively simple relationship. The system description in this method consists of two parts:

1 Finding the equation(s) relating the state variables to the input(s) (**the state equation**).

2 Finding the output variables in terms of the state variables (**the output equation**).

The analysis procedure, therefore, consists of solving the state equation first, and then solving the output equation. The state space description is capable of determining every possible system variable (or output) from the knowledge of the input and the initial state (conditions) of the system. For this reason it is an *internal description* of the system.

By its nature, the state variable analysis is eminently suited for multiple-input, multiple-output (MIMO) systems. In addition, the state-space techniques are useful for several other reasons, including the following:

1. Time-varying parameter systems and nonlinear systems can be characterized effectively with state-space descriptions.
2. State equations lend themselves readily to accurate simulation on analog or digital computers.
3. For second-order systems ($n = 2$), a graphical method called **phase-plane analysis** can be used on state equations, whether they are linear or nonlinear.

4. State equations can yield a great deal of information about a system even when they are not solved explicitly.

This chapter requires some understanding of matrix algebra. Section B.6 is a self-contained treatment of matrix algebra, which should be more than adequate for the purposes of this chapter.

13.1 Introduction

From the discussion in Chapter 1, we know that to determine a system's response(s) at any instant t, we need to know the system's inputs during its entire past, from $-\infty$ to t. If the inputs are known only for $t > t_0$, we can still determine the system output(s) for any $t > t_0$, provided we know certain initial conditions in the system at $t = t_0$. These initial conditions collectively are called the **initial state** of the system (at $t = t_0$).

The state of a system at any instant t_0 is the smallest set of numbers $x_1(t_0)$, $x_2(t_0)$, ..., $x_n(t_0)$ which is sufficient to determine the behavior of the system for all time $t > t_0$ when the input(s) to the system is known for $t > t_0$. The variables $x_1, x_2, \ldots, x_n$ are known as **state variables.**

The initial conditions of a system can be specified in many different ways. Consequently, the system state can also be specified in many different ways. This means that state variables are not unique. The concept of a system state is very important. We know that an output $y(t)$ at any instant $t > t_0$ can be determined from the initial state $\{x(t_0)\}$ and a knowledge of the input $f(t)$ during the interval (t_0, t). Therefore, the output $y(t_0)$ (at $t = t_0$) is determined from the initial state $\{x(t_0)\}$ and the input $f(t)$ during the interval (t_0, t_0). The latter is $f(t_0)$. Hence, the output at any instant is determined completely from a knowledge of the system state and the input at that instant. This result is also valid for multiple-input, multiple-output (MIMO) systems, where every possible system output at any instant t is determined completely from a knowledge of the system state and the input(s) at the instant t. These ideas should become clear from the following example of an RLC circuit.

■ **Example 13.1**

Find a state-space description of the RLC circuit shown in Fig. 13.1. Verify that all possible system outputs at some instant t can be determined from a knowledge of the system state and the input at that instant t.

It is known that inductor currents and capacitor voltages in an RLC circuit can be used as one possible choice of state variables. For this reason, we shall choose x_1 (the capacitor voltage) and x_2 (the inductor current) as our state variables.

The node equation at the intermediate node is

$$i_3 = i_1 - i_2 - x_2$$

but $i_3 = 0.2\dot{x}_1$, $i_1 = 2(f - x_1)$, $i_2 = 3x_1$. Hence

$$0.2\dot{x}_1 = 2(f - x_1) - 3x_1 - x_2$$

or

$$\dot{x}_1 = -25x_1 - 5x_2 + 10f$$

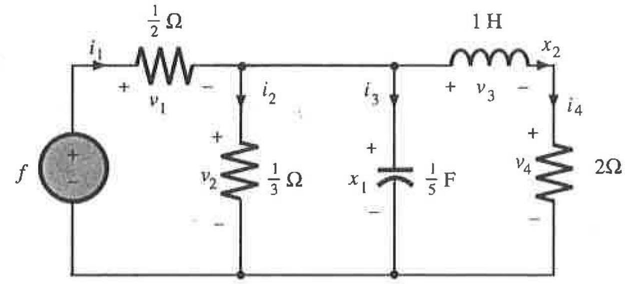


Fig. 13.1 *RLC* network for Example 13.1.

This is the first state equation. To obtain the second state equation, we sum the voltages in the extreme right loop formed by C, L, and the 2 Ω resistor so that they are equal to zero:

$$-x_1 + \dot{x}_2 + 2x_2 = 0$$

or

$$\dot{x}_2 = x_1 - 2x_2$$

Thus, the two state equations are

$$\dot{x}_1 = -25x_1 - 5x_2 + 10f \tag{13.1a}$$

$$\dot{x}_2 = x_1 - 2x_2 \tag{13.1b}$$

Every possible output can now be expressed as a linear combination of x_1, x_2, and f. From Fig. 13.1, we have

$$\begin{aligned}
v_1 &= f - x_1 \\
i_1 &= 2(f - x_1) \\
v_2 &= x_1 \\
i_2 &= 3x_1 \\
i_3 &= i_1 - i_2 - x_2 = 2(f - x_1) - 3x_1 - x_2 = -5x_1 - x_2 + 2f \\
i_4 &= x_2 \\
v_4 &= 2i_4 = 2x_2 \\
v_3 &= x_1 - v_4 = x_1 - 2x_2
\end{aligned} \tag{13.2}$$

This set of equations is known as the **output equation** of the system. It is clear from this set that every possible output at some instant t can be determined from a knowledge of $x_1(t)$, $x_2(t)$, and $f(t)$, the system state and the input at the instant t. Once we solve the state equations (13.1) to obtain $x_1(t)$ and $x_2(t)$, we can determine every possible output for any given input $f(t)$. ■

If we already have a system equation in the form of an nth-order differential equation, we can convert it into a state equation as follows. Consider the system equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = f(t) \tag{13.3}$$

One possible set of initial conditions is $y(0)$, $\dot{y}(0)$, ... , $y^{(n-1)}(0)$. Let us define y, $\dot{y}$, $\ddot{y}$, ... , $y^{(n-1)}$ as the state variables and, for convenience, let us rename the n state variables as x_1, x_2, ... , x_n:

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \\ x_3 &= \ddot{y} \\ &\vdots \\ x_n &= y^{(n-1)} \end{aligned} \tag{13.4}$$

According to Eq. (13.4), we have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned}$$

and, according to Eq. (13.3),

$$\dot{x}_n = -a_{n-1}x_n - a_{n-2}x_{n-1} - \cdots - a_1x_2 - a_0x_1 + f \tag{13.5a}$$

These n simultaneous first-order differential equations are the state equations of the system. The output equation is

$$y = x_1 \tag{13.5b}$$

For continuous-time systems, the state equations are n simultaneous first-order differential equations in n state variables x_1, x_2, ... , x_n of the form

$$\dot{x}_i = g_i(x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_j) \qquad i = 1, 2, \ldots, n$$

where f_1, f_2, ... , f_n are the j system inputs. For a linear system, these equations reduce to a simpler linear form

$$\dot{x}_k = a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n + b_{k1}f_1 + b_{k2}f_2 + \cdots + b_{kj}f_j \qquad k = 1, 2, \ldots, n \tag{13.6a}$$

and the output equations are of the form

$$y_m = c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mn}x_n + d_{m1}f_1 + d_{m2}f_2 + \cdots + d_{mj}f_j \qquad m = 1, 2, \ldots, k \tag{13.6b}$$

The set of Equations (13.6a) and (13.6b) is called a **dynamical** equation. When it is used to describe a system, it is called the **dynamical-equation description** or **state-variable** description of the system. The n simultaneous first-order state equations are also known as the **normal-form** equations.

These equations can be written more conveniently in matrix form:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_n \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} \\ b_{21} & b_{22} & \cdots & b_{2j} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ \cdots \\ f_j \end{bmatrix}}_{\mathbf{f}} \tag{13.7a}$$

and

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_k \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{k1} & c_{k2} & \cdots & c_{kn} \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1j} \\ d_{21} & d_{22} & \cdots & d_{2j} \\ \cdots & \cdots & \cdots & \cdots \\ d_{k1} & d_{k2} & \cdots & d_{kj} \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ \cdots \\ f_j \end{bmatrix}}_{\mathbf{f}} \tag{13.7b}$$

or

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bf} \tag{13.8a}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Df} \tag{13.8b}$$

Equation (13.8a) is the state equation and Eq. (13.8b) is the output equation. The vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{f}$ are the state vector, the output vector, and the input vector, respectively.

For discrete-time systems, the state equations are n simultaneous first-order difference equations. Discrete-time systems are discussed in Sec. 13.6.

13.2 A Systematic Procedure for Determining State Equations

We shall discuss here a systematic procedure for determining the state-space description of linear time-invariant systems. In particular, we shall consider two types of systems: (1) *RLC* networks and (2) systems specified by block diagrams or nth-order transfer functions.

13.2-1 Electrical Circuits

The method used in Example 13.1 proves effective in most of the simple cases. The steps are as follows:

1. Choose all independent capacitor voltages and inductor currents to be the state variables.

2. Choose a set of loop currents; express the state variables and their first derivatives in terms of these loop currents.

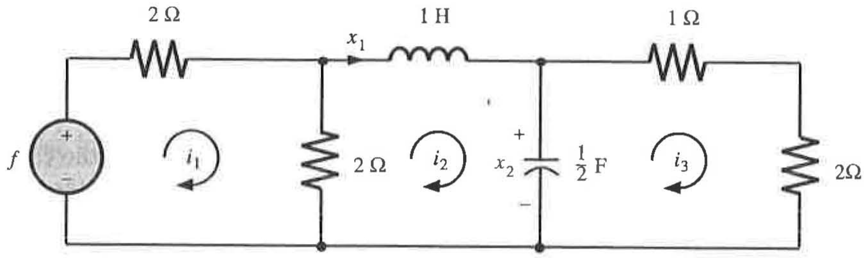


Fig. 13.2 *RLC* network for Example 13.2.

3. Write the loop equations and eliminate all variables other than state variables (and their first derivatives) from the equations derived in Steps 2 and 3.

■ **Example 13.2**

Write the state equations for the network shown in Fig. 13.2.

Step 1. There is one inductor and one capacitor in the network. Therefore, we shall choose the inductor current x_1 and the capacitor voltage x_2 as the state variables.

Step 2. The relationship between the loop currents and the state variables can be written by inspection:

$$x_1 = i_2 \tag{13.9a}$$

$$\tfrac{1}{2}\dot{x}_2 = i_2 - i_3 \tag{13.9b}$$

Step 3. The loop equations are

$$4i_1 - 2i_2 = f \tag{13.10a}$$

$$2(i_2 - i_1) + \dot{x}_1 + x_2 = 0 \tag{13.10b}$$

$$-x_2 + 3i_3 = 0 \tag{13.10c}$$

Now we eliminate i_1, i_2, and i_3 from Eqs. (13.9) and (13.10) as follows. From Eq. (13.10b), we have

$$\dot{x}_1 = 2(i_1 - i_2) - x_2$$

We can eliminate i_1 and i_2 from this equation by using Eqs. (13.9a) and (13.10a) to obtain

$$\dot{x}_1 = -x_1 - x_2 + \tfrac{1}{2}f$$

The substitution of Eqs. (13.9a) and (13.10c) in Eq. (13.9b) yields

$$\dot{x}_2 = 2x_1 - \tfrac{2}{3}x_2$$

These are the desired state equations. We can express them in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} f \tag{13.11}$$

The derivation of state equations from loop equations is facilitated considerably by choosing loops in such a way that only one loop current passes through each of the inductors or capacitors. ■

An Alternative procedure

We can also determine the state equations by the following procedure:

1. Choose all independent capacitor voltages and inductor currents to be the state variables.

2. Replace each capacitor by a fictitious voltage source equal to the capacitor voltage, and replace each inductor by a fictitious current source equal to the inductor current. This step will transform the RLC network into a network consisting only of resistors, current sources, and voltage sources.

3. Find the current through each capacitor and equate it to $C\dot{x}_i$, where x_i is the capacitor voltage. Similarly, find the voltage across each inductor and equate it to $L\dot{x}_j$, where x_j is the inductor current.

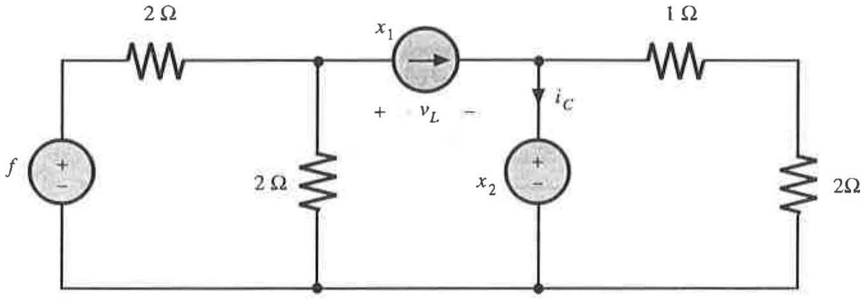


Fig. 13.3 Equivalent circuit of the network in Fig. 13.2.

■ **Example 13.3**

Using the above procedure, write the state equations for the network in Fig. 13.2.

In the network in Fig. 13.2, we replace the inductor by a current source of current x_1 and the capacitor by a voltage source of voltage x_2, as shown in Fig. 13.3. The resulting network consists of five resistors, two voltage sources, and one current source. We can determine the voltage v_L across the inductor and the current i_c through the capacitor by using the principle of superposition. This step can be accomplished by inspection. For example, v_L has three components arising from three sources. To compute the component due to f, we assume that $x_1 = 0$ (open circuit) and $x_2 = 0$ (short circuit). Under these conditions, all of the network to the right of the 2 Ω resistor is opened, and the component of v_L due to f is the voltage across the 2 Ω resistor. This voltage is clearly $\frac{1}{2}f$. Similarly, to find the component of v_L due to x_1, we short f and x_2. The source x_1 sees an equivalent resistor of 1Ω across it, and hence $v_L = -x_1$. Continuing the process, we find that the component of v_L due to x_2 is $-x_2$. Hence

$$v_L = \dot{x}_1 = \frac{1}{2}f - x_1 - x_2 \tag{13.12a}$$

Using the same procedure, we find

$$i_c = \frac{1}{2}\dot{x}_2 = x_1 - \frac{1}{3}x_2 \tag{13.12b}$$

These equations are identical to the state equations (13.11) obtained earlier.† ■

13.2-2 State Equations From Transfer Function

It is relatively easy to determine the state equations of a system specified by its transfer function. Consider, for example, a first-order system with the transfer function

$$H(s) = \frac{1}{s+a} \tag{13.13}$$

The system realization appears in Fig. 13.4. The integrator output serves as a natural state variable since, in practical realization, initial conditions are placed on the integrator output. From Fig. 13.4, we have

$$\dot{x} = -ax + f \tag{13.14a}$$

$$y = x \tag{13.14b}$$

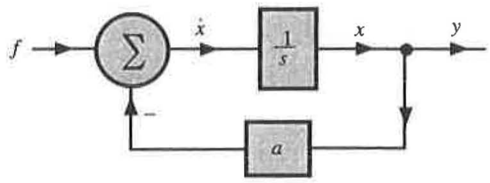


Fig. 13.4

In Sec. 6.6 we saw that a given transfer function can be realized in several ways. Consequently, we should be able to obtain different state-space descriptions of the same system by using different realizations. This assertion will be clarified by the following example.

■ **Example 13.4**

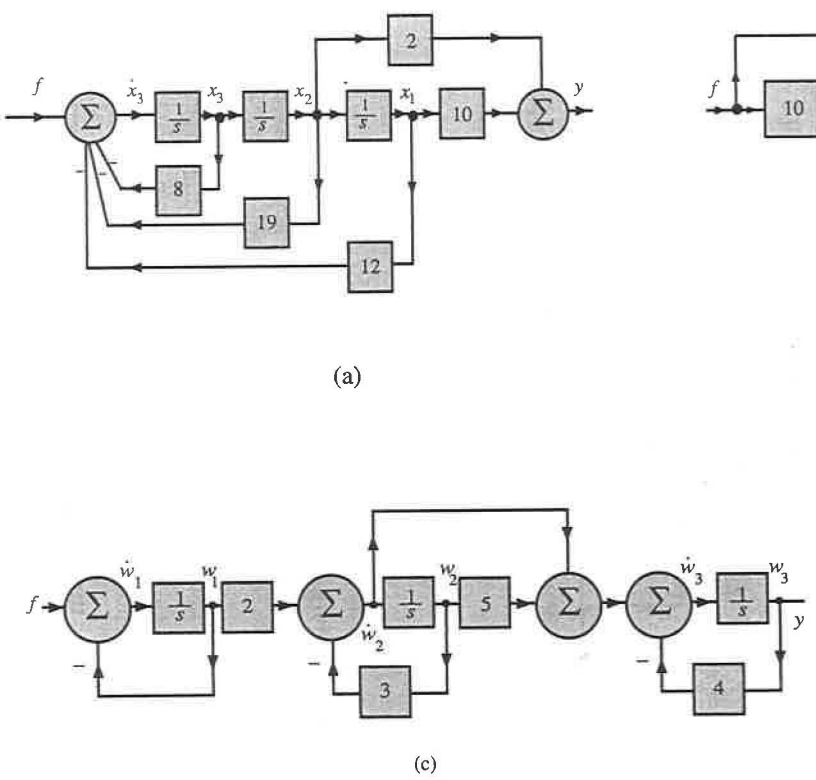
Determine the state-space description of a system specified by the transfer function

$$H(s) = \frac{2s+10}{s^3+8s^2+19s+12} \tag{13.15a}$$

$$= \left(\frac{2}{s+1}\right)\left(\frac{s+5}{s+3}\right)\left(\frac{1}{s+4}\right) \tag{13.15b}$$

$$= \frac{\frac{4}{3}}{s+1} - \frac{2}{s+3} + \frac{\frac{2}{3}}{s+4} \tag{13.15c}$$

†This procedure requires modification if the system contains all-capacitor voltage source tie sets or all-inductor current source cut sets. In the case of all-capacitor voltage source tie sets, all capacitor voltages cannot be independent. One capacitor voltage can be expressed in terms of the remaining capacitor voltages and the voltage source(s) in that tie set. Consequently, one of the capacitor voltages should not be used as a state variable, and that capacitor should not be replaced by a voltage source. Similarly, in all-inductor current source tie sets, one inductor should not be replaced by a current source. If there are all-capacitor tie sets or all-inductor cut sets only, no further complications occur. In all-capacitor-voltage source tie sets and/or all-inductor-current source cut sets, we have additional difficulties in that the terms involving derivatives of the input may occur. This problem can be solved by redefining the state variables. The final state variables will not be capacitor voltages and inductor currents.



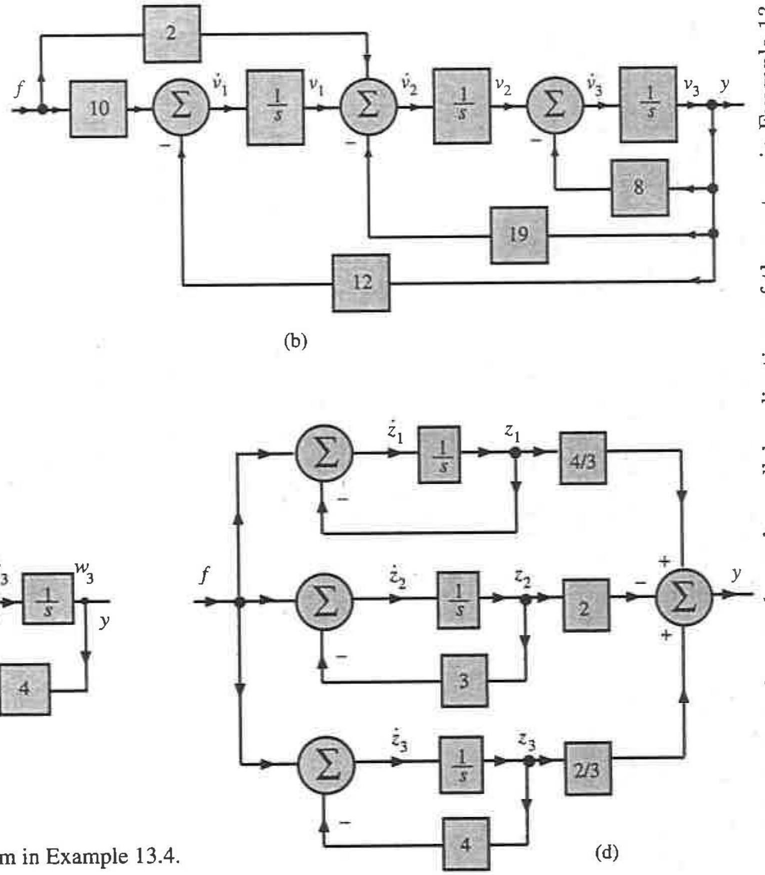


Fig. 13.5 Canonical, cascade, and parallel realizations of the system in Example 13.4.

Fig. 13.5 Canonical, cascade, and parallel realizations of the system in Example 13.4.

Using the procedure developed in Sec. 6.6, we shall realize $H(s)$ in Eq. (13.15) with four different realizations: (i) the controller canonical form [Eq. (13.15a)], (ii) the observer canonical form [Eq. (13.15a)], (iii) cascade realization [Eq. (13.15b)] and (iv) parallel realization [Eq. (13.15c)]. These realizations are depicted in Figs. 13.5a, 13.5b, 13.5c, and 13.5d, respectively. As mentioned earlier, the output of each integrator serves as a natural state variable.

1. Canonical Forms

Here we shall realize the system using the first (controller) canonical form discussed in Sec. 6.6-1. If we choose the state variables to be the three integrator outputs x_1, x_2, and x_3, then, according to Fig. 13.5a,

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= -12x_1 - 19x_2 - 8x_3 + f\end{aligned} \tag{13.16a}$$

Also, the output y is given by

$$y = 10x_1 + 2x_2 \tag{13.16b}$$

Equations (13.16a) are the state equations, and Eq. (13.16b) is the output equation. In matrix form we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -12 & -19 & -8\end{bmatrix}}_{\mathbf{A}}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}}_{\mathbf{B}} f \tag{13.17a}$$

and

$$\mathbf{y} = \underbrace{[10 \quad 2 \quad 0]}_{\mathbf{C}}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \tag{13.17b}$$

We can also realize $H(s)$ by using the second (observer) canonical form (discussed in Appendix 6.1), as shown in Fig. 13.5b. If we label the output of the three integrators from left to right as the state variables v_1, v_2, and v_3, then, according to Fig. 13.5b,

$$\begin{aligned}\dot{v}_1 &= -12v_3 + 10f\\ \dot{v}_2 &= v_1 - 19v_3 + 2f\\ \dot{v}_3 &= v_2 - 8v_3\end{aligned} \tag{13.18a}$$

and the output y is given by

$$y = v_3 \tag{13.18b}$$

Hence

$$\begin{bmatrix}\dot{v}_1\\ \dot{v}_2\\ \dot{v}_3\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 0 & -12\\ 1 & 0 & -19\\ 0 & 1 & -8\end{bmatrix}}_{\hat{\mathbf{A}}}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} + \underbrace{\begin{bmatrix}10\\ 2\\ 0\end{bmatrix}}_{\hat{\mathbf{B}}} f \tag{13.19a}$$

and

$$\mathbf{y} = \underbrace{[0 \quad 0 \quad 1]}_{\hat{\mathbf{C}}} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{13.19b}$$

Observe closely the relationship between the state-space descriptions of $H(s)$ that use the controller canonical form [Eqs. (13.17)] and those using the observer canonical form [Eqs. (13.19)]. The $\mathbf{A}$ matrices in these two cases are the transpose of one another; also, the $\mathbf{B}$ of one is the transpose of $\mathbf{C}$ in the other, and vice versa. Hence

$$\begin{aligned}(\mathbf{A})^T &= \hat{\mathbf{A}} \\ (\mathbf{B})^T &= \hat{\mathbf{C}} \\ (\mathbf{C})^T &= \hat{\mathbf{B}}\end{aligned} \tag{13.20}$$

This is no coincidence. This duality relation is generally true.[1]

2. Series Realization

The three integrator outputs w_1, w_2, and w_3 in Fig. 13.5c are the state variables. The state equations are

$$\dot{w}_1 = -w_1 + f \tag{13.21a}$$

$$\dot{w}_2 = 2w_1 - 3w_2 \tag{13.21b}$$

$$\dot{w}_3 = 5w_2 + \dot{w}_2 - 4w_3 \tag{13.21c}$$

and the output equation is

$$y = w_3$$

The elimination of $\dot{w}_2$ from Eq. (13.21c) by using Eq. (13.21b) converts these equations into the desired state form

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} f \tag{13.22a}$$

and

$$y = [0 \quad 0 \quad 1] \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{13.22b}$$

3. Parallel Realization (Diagonal Representation)

The three integrator outputs z_1, z_2, and z_3 in Fig. 13.5d are the state variables. The state equations are

$$\begin{aligned}\dot{z}_1 &= -z_1 + f \\ \dot{z}_2 &= -3z_2 + f \\ \dot{z}_3 &= -4z_3 + f\end{aligned} \tag{13.23a}$$

and the output equation is

$$y = \tfrac{4}{3}z_1 - 2z_2 + \tfrac{2}{3}z_3 \tag{13.23b}$$

Therefore, the equations in the matrix form are

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} f \tag{13.24a}$$

$$y = \begin{bmatrix} \tfrac{4}{3} & -2 & \tfrac{2}{3} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{13.24b}$$

■

⊙ **Computer Example C13.1**
Solve Example 13.4 using MATLAB.

Caution: The convention of MATLAB for labeling state variables $x_1, x_2, \ldots, x_n$ in a block diagram, such as shown in Fig. 13.5a, is reversed. What we label x_1 is x_n, and x_2 is x_{n-1}, and so on.

```
num=[2 10]; den=[1 8 19 12];
[A,B,C,D]=tf2ss(num,den)
% In order to find the transfer function from A, B, C, and D, use
[num, den]=ss2tf(A,B,C,D)
printsys(num,den)
```
⊙

A General Case

It is clear that a system has several state-space descriptions. Notable among these are the canonical-form variables and the diagonalized variables (in the parallel realization). State equations in these forms can be written immediately by inspection of the transfer function. Consider the general nth-order transfer function

$$H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{13.25a}$$

$$= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)}$$

$$= \frac{k_1}{s-\lambda_1} + \frac{k_2}{s-\lambda_2} + \cdots + \frac{k_n}{s-\lambda_n} \tag{13.25b}$$

Figures 13.6a and 13.6b show the realizations of $H(s)$, using the controller canonical form [Eq. (13.25a)] and the parallel form [Eq. (13.25b)], respectively.

The n integrator outputs $x_1, x_2, \ldots, x_n$ in Fig. 13.6a are the state variables. It is clear that

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdots\cdots\cdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_{n-1}x_n - a_{n-2}x_{n-1} - \cdots - a_1 x_2 - a_0 x_1 + f \end{aligned} \tag{13.26a}$$

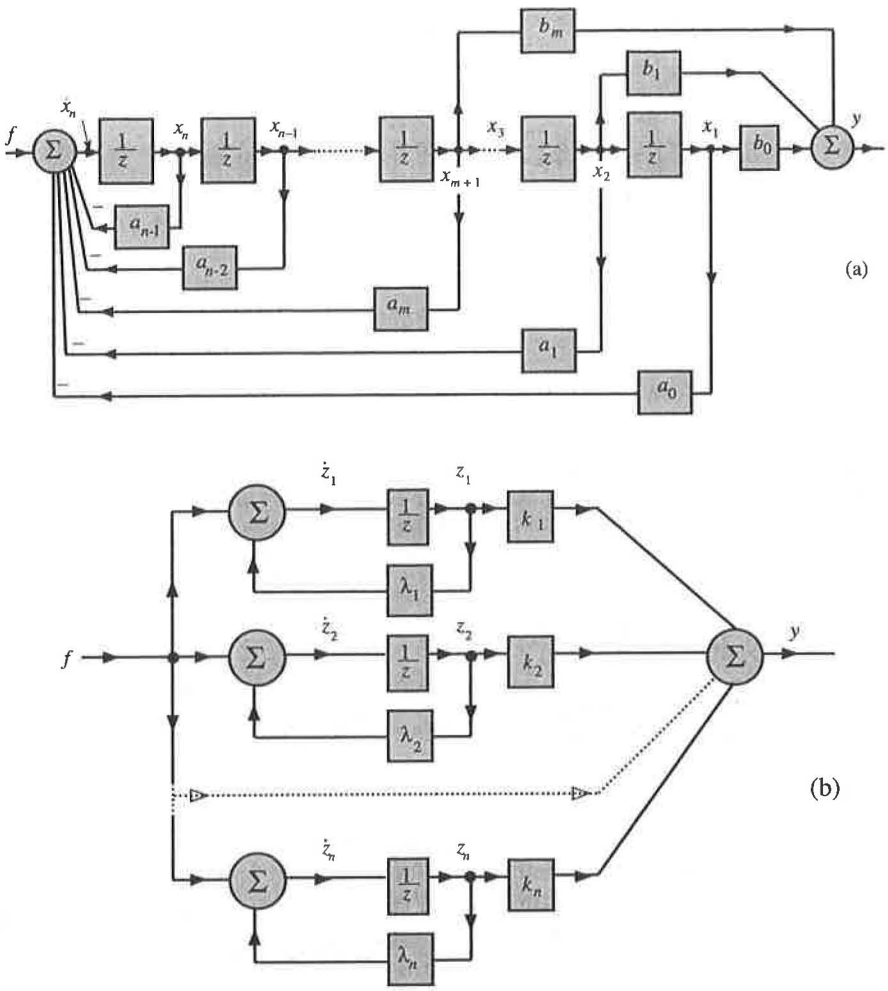


Fig. 13.6 Controller canonical and parallel realizations for an nth order LTIC system.

and output y is

$$y = b_0 x_1 + b_1 x_2 + \cdots + b_m x_{m+1} \tag{13.26b}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix} f \tag{13.27a}$$

and

$$y = [b_0 \quad b_1 \quad \cdots \quad b_m \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{13.27b}$$

Observe that these equations (state equations and output equation) can be written immediately by inspection of $H(s)$.

The n integrator outputs $z_1, z_2, \ldots, z_n$ in Fig. 13.6b are the state variables. It is clear that

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 + f \\ \dot{z}_2 &= \lambda_2 z_2 + f \\ &\cdots\cdots\cdots\cdots \\ \dot{z}_n &= \lambda_n z_n + f \end{aligned} \tag{13.28a}$$

and

$$y = k_1 z_1 + k_2 z_2 + \cdots + k_n z_n \tag{13.28b}$$

or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \cdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ z_{n-1} \\ z_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \\ 1 \end{bmatrix} f \tag{13.29a}$$

and

$$y = [k_1 \quad k_2 \quad \cdots \quad k_{n-1} \quad k_n] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} \tag{13.29b}$$

The state equation (13.29a) and the output equation (13.29b) can be written immediately by inspection of the transfer function $H(s)$ in Eq. (13.25b). Observe that the diagonalized form of the state matrix [Eq. (13.29a)] has the transfer function poles as its diagonal elements. The presence of repeated poles in $H(s)$ will modify the procedure slightly. The handling of these cases is discussed in Sec. 6.6.

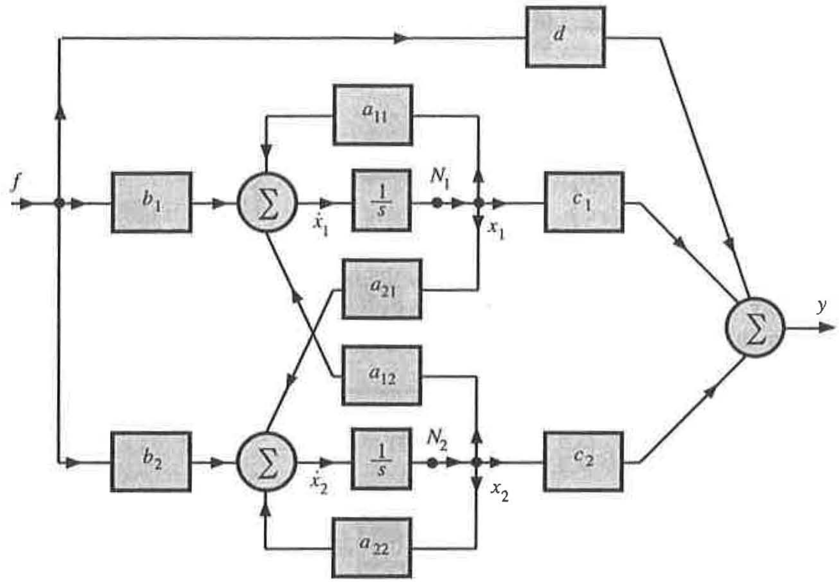


Fig. 13.7 Realization of a second-order system.

It is clear from the above discussion that a state-space description is not unique. For any realization of $H(s)$ using integrators, scalar multipliers, and adders, a corresponding state-space description exists. Since there are many possible realizations of $H(s)$, there are many possible state-space descriptions.

Realization

Consider a second-order system with a single input f, a single output y, and two state variables, x_1 and x_2. The system equations are

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_1 f$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_2 f \tag{13.30a}$$

and

$$y = c_1 x_1 + c_2 x_2 + df \tag{13.30b}$$

Figure 13.7 shows the block diagram of the realized system. The initial conditions $x_1(0)$ and $x_2(0)$ should be applied at N_1 and N_2. This procedure can be easily extended to general multiple-input, multiple-output systems with n state variables.

13.3 Solution of State Equations

The state equations of a linear system are n simultaneous linear differential equations of the first order. We studied the techniques of solving linear differential equations in Chapters 2 and 6. The same techniques can be applied to state equations without any modification. However, it is more convenient to carry out the solution in the framework of matrix notation.

These equations can be solved in both the time domain and frequency domain (Laplace transform). The latter requires fewer new concepts and is therefore easier to deal with than the time-domain solution. For this reason, we shall first consider the Laplace transform solution.

13.3-1 Laplace Transform Solution of State Equations

The kth state equation [Eq. (13.6a)] is of the form

$$\dot{x}_k = a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n + b_{k1}f_1 + b_{k2}f_2 + \cdots + b_{kj}f_j \tag{13.31a}$$

We shall take the Laplace transform of this equation. Let

$$x_k(t) \Longleftrightarrow X_k(s)$$

so that

$$\dot{x}_k(t) \Longleftrightarrow sX_k(s) - x_k(0)$$

Also, let

$$f_i(t) \Longleftrightarrow F_i(s)$$

The Laplace transform of Eq. (13.31a) yields

$$sX_k(s) - x_k(0) = a_{k1}X_1(s) + a_{k2}X_2(s) + \cdots + a_{kn}X_n(s) + b_{k1}F_1(s) + b_{k2}F_2(s) + \cdots + b_{kj}F_j(s) \tag{13.31b}$$

Taking the Laplace transforms of all n state equations, we obtain

$$s\underbrace{\begin{bmatrix} X_1(s) \\ X_2(s) \\ \cdots \\ X_n(s) \end{bmatrix}}_{\mathbf{X}(s)} - \underbrace{\begin{bmatrix} x_1(0) \\ x_2(0) \\ \cdots \\ x_n(0) \end{bmatrix}}_{\mathbf{x}(0)} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} X_1(s) \\ X_2(s) \\ \cdots \\ X_n(s) \end{bmatrix}}_{\mathbf{X}(s)} + \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} \\ b_{21} & b_{22} & \cdots & b_{2j} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} F_1(s) \\ F_2(s) \\ \cdots \\ F_j(s) \end{bmatrix}}_{\mathbf{F}(s)} \tag{13.32a}$$

Defining the vectors, as indicated above, we have

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{AX}(s) + \mathbf{BF}(s)$$

or

$$s\mathbf{X}(s) - \mathbf{AX}(s) = \mathbf{x}(0) + \mathbf{BF}(s)$$

and

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{F}(s) \tag{13.32b}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. From Eq. 13.32b, we have

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\mathbf{F}(s)] \tag{13.33a}$$

$$= \mathbf{\Phi}(s)[\mathbf{x}(0) + \mathbf{B}\mathbf{F}(s)] \tag{13.33b}$$

where

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \tag{13.34}$$

Thus, from Eq. (13.33b),

$$\mathbf{X}(s) = \mathbf{\Phi}(s)\mathbf{x}(0) + \mathbf{\Phi}(s)\mathbf{B}\mathbf{F}(s) \tag{13.35a}$$

and

$$\mathbf{x}(t) = \underbrace{\mathcal{L}^{-1}[\mathbf{\Phi}(s)]\mathbf{x}(0)}_{\text{zero-input component}} + \underbrace{\mathcal{L}^{-1}[\mathbf{\Phi}(s)\mathbf{B}\mathbf{F}(s)]}_{\text{zero-state component}} \tag{13.35b}$$

Equation (13.35b) gives the desired solution. Observe the two components of the solution. The first component yields $\mathbf{x}(t)$ when the input $f(t) = 0$. Hence the first component is the zero-input component. In a similar manner, we see that the second component is the zero-state component.

Example 13.5

Find the state vector $\mathbf{x}(t)$ for the system whose state equation is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f}$$

where

$$\mathbf{A} = \begin{bmatrix} -12 & \frac{2}{3} \\ -36 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \qquad \mathbf{f}(t) = u(t)$$

and the initial conditions are $x_1(0) = 2$, $x_2(0) = 1$.

From Eq. (13.33b), we have

$$\mathbf{X}(s) = \mathbf{\Phi}(s)[\mathbf{x}(0) + \mathbf{B}\mathbf{F}(s)]$$

Let us first find $\mathbf{\Phi}(s)$. We have

$$(s\mathbf{I} - \mathbf{A}) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -12 & \frac{2}{3} \\ -36 & -1 \end{bmatrix} = \begin{bmatrix} s+12 & -\frac{2}{3} \\ 36 & s+1 \end{bmatrix}$$

and

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} \frac{s+1}{(s+4)(s+9)} & \frac{2/3}{(s+4)(s+9)} \\ \frac{-36}{(s+4)(s+9)} & \frac{s+12}{(s+4)(s+9)} \end{bmatrix} \tag{13.36a}$$

Now, $\mathbf{x}(0)$ is given as

$$\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Also, $F(s) = \frac{1}{s}$, and

$$\mathbf{B}\mathbf{F}(s) = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \frac{1}{s} = \begin{bmatrix} \frac{1}{3s} \\ \frac{1}{s} \end{bmatrix}$$

Therefore

$$\mathbf{x}(0)+\mathbf{BF}(s)=\begin{bmatrix}2+\frac{1}{3s}\\ 1+\frac{1}{s}\end{bmatrix}=\begin{bmatrix}\frac{6s+1}{3s}\\ \frac{s+1}{s}\end{bmatrix}$$

and

$$\mathbf{X}(s)=\mathbf{\Phi}(s)[\mathbf{x}(0)+\mathbf{BF}(s)]$$

$$=\begin{bmatrix}\frac{s+1}{(s+4)(s+9)} & \frac{\frac{2}{3}}{(s+4)(s+9)}\\ \frac{-36}{(s+4)(s+9)} & \frac{s+12}{(s+4)(s+9)}\end{bmatrix}\begin{bmatrix}\frac{6s+1}{3s}\\ \frac{s+1}{s}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{2s^2+3s+1}{s(s+4)(s+9)}\\ \frac{s-59}{(s+4)(s+9)}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{\frac{1}{36}}{s}-\frac{\frac{21}{20}}{s+4}+\frac{\frac{136}{45}}{s+9}\\ \frac{-\frac{63}{5}}{s+4}+\frac{\frac{68}{5}}{s+9}\end{bmatrix}$$

The inverse Laplace transform of this equation yields

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}\left(\frac{1}{36}-\frac{21}{20}e^{-4t}+\frac{136}{45}e^{-9t}\right)u(t)\\ \left(-\frac{63}{5}e^{-4t}+\frac{68}{5}e^{-9t}\right)u(t)\end{bmatrix} \qquad (13.36b)$$

■

⊙ **Computer Example C13.2**
Solve Example 13.5 using MATLAB.
Caution: See caution in Example C13.1.

```
A=[-12 2/3;-36 -1]; B=[1/3; 1];
C=[0 0]; D=0;
x0=[2;1];
t=0:.01:3; t=t';
f=ones(length(t),1);
[y,x]=lsim(A,B,C,D,f,t,x0);
plot(t,x)   ⊙
```

The Output

The output equation is given by

$$\mathbf{y}=\mathbf{Cx}+\mathbf{Df}$$

and

$$\mathbf{Y}(s)=\mathbf{CX}(s)+\mathbf{DF}(s)$$

The substitution of Eq. (13.33b) into this equation yields

$$\mathbf{Y}(s) = \mathbf{C}\{\mathbf{\Phi}(s)[\mathbf{x}(0) + \mathbf{BF}(s)]\} + \mathbf{DF}(s)$$
$$= \underbrace{\mathbf{C\Phi}(s)\mathbf{x}(0)}_{\text{zero-input response}} + \underbrace{[\mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D}]\mathbf{F}(s)}_{\text{zero-state response}} \tag{13.37}$$

The zero-state response (that is, the response $\mathbf{Y}(s)$ when $\mathbf{x}(0)=\mathbf{0}$), is given by

$$\mathbf{Y}(s) = [\mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D}]\mathbf{F}(s) \tag{13.38a}$$

Note that the transfer function of a system is defined under the zero-state condition [see Eq. (6.53)]. The matrix $\mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D}$ is the **transfer function matrix** $\mathbf{H}(s)$ of the system, which relates the responses $y_1, y_2, \dots, y_k$ to the inputs $f_1, f_2, \dots, f_j$:

$$\mathbf{H}(s) = \mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D} \tag{13.38b}$$

and the zero-state response is

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{F}(s) \tag{13.39}$$

The matrix $\mathbf{H}(s)$ is a $k \times j$ matrix (k is the number of outputs and j is the number of inputs). The ijth element $H_{ij}(s)$ of $H(s)$ is the transfer function that relates the output $y_i(t)$ to the input $f_j(t)$.

■ **Example 13.6**

Let us consider a system with a state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{13.40a}$$

and an output equation

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{13.40b}$$

In this case,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13.40c}$$

and

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \tag{13.41}$$

Hence, the transfer function matrix $\mathbf{H}(s)$ is given by

$$\mathbf{H}(s) = \mathbf{C}\mathbf{\Phi}(s)\mathbf{B} + \mathbf{D}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+4}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{s+4}{s+2} & \frac{1}{s+2} \\ \frac{2(s-2)}{(s+1)(s+2)} & \frac{s^2+5s+2}{(s+1)(s+2)} \end{bmatrix} \tag{13.42}$$

and the zero-state response is

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{F}(s)$$

Remember that the ijth element of the transfer function matrix in Eq. (13.42) represents the transfer function that relates the output $y_i(t)$ to the input $f_j(t)$. Thus, the transfer function that relates the output y_3 to the input f_2 is $H_{32}(s)$, where

$$H_{32}(s) = \frac{s^2 + 5s + 2}{(s+1)(s+2)} \qquad \blacksquare$$

⊙ **Computer Example C13.3**

Solve Example 13.6 using MATLAB.

Caution: The common factor $(s+1)$ in two of the transfer functions in Eq. (13.42) are canceled. The MATLAB answer gives transfer function with common factor.

```
A=[0 1;-2 -3]; B=[1 0;1 1];
C=[1 0;1 1;0 2]; D=[0 0;1 0;0 1];
[num1,den1]=ss2tf(A,B,C,D,1)
[num2,den2]=ss2tf(A,B,C,D,2)
```
⊙

Characteristic Roots (Eigenvalues) of a Matrix

It is interesting to observe that the denominator of every transfer function in Eq. (13.42) is $(s+1)(s+2)$ with the exception of $H_{21}(s)$ and $H_{22}(s)$, where the cancellation of the factor $(s+1)$ occurs. This fact is no coincidence. We see that the denominator of every element of $\mathbf{\Phi}(s)$ is $|s\mathbf{I}-\mathbf{A}|$ because $\mathbf{\Phi}(s) = (s\mathbf{I}-\mathbf{A})^{-1}$, and the inverse of a matrix has its determinant in the denominator. Since **C**, **B**, and **D** are matrices with constant elements, we see from Eq. (13.38b) that the denominator of $\mathbf{\Phi}(s)$ will also be the denominator of $\mathbf{H}(s)$. Hence, the denominator of every element of $\mathbf{H}(s)$ is $|s\mathbf{I}-\mathbf{A}|$, except for the possible cancellation of the common factors mentioned earlier. In other words, the poles of all transfer functions of the system are also the zeros of the polynomial $|s\mathbf{I}-\mathbf{A}|$. *Therefore, the zeros of the polynomial* $|s\mathbf{I}-\mathbf{A}|$ *are the characteristic roots of the system.* Hence, the characteristic roots of the system are the roots of the equation

$$|s\mathbf{I} - \mathbf{A}| = 0 \tag{13.43a}$$

Since $|s\mathbf{I} - \mathbf{A}|$ is an nth-order polynomial in s with n zeros $\lambda_1, \lambda_2, \ldots, \lambda_n$, we can write Eq. (13.43a) as

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_{n-1}s^{n+1} + \cdots + a_1 s + a_0$$
$$= (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n) = 0 \tag{13.43b}$$

For the system in Example 13.6,

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s & 0 \\ 0 & s \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix}$$
$$= \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix}$$
$$= s^2 + 3s + 2 \tag{13.44a}$$
$$= (s+1)(s+2) \tag{13.44b}$$

Hence

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2$$

Equation (13.43) is known as the **characteristic equation of the matrix A**, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic roots of $\mathbf{A}$. The term **eigenvalue**, meaning "characteristic value" in German, is also commonly used in the literature. Thus, we have shown that the characteristic roots of a system are the eigenvalues (characteristic values) of the matrix $\mathbf{A}$.

At this point, the reader will recall that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the poles of the transfer function, then the zero-input response is of the form

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t} \tag{13.45}$$

This fact is also obvious from Eq. (13.38). The denominator of every element of the zero-input response matrix $\mathbf{C}\mathbf{\Phi}(s)\mathbf{x}(0)$ is $|s\mathbf{I} - \mathbf{A}| = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n)$. Therefore, the partial-fraction expansion and the subsequent inverse Laplace transform will yield a zero-input component of the form in Eq. (13.45).

13.3-2 Time-Domain Solution of State Equations

The state equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f} \tag{13.46}$$

We now show that the solution of the vector differential Equation (13.46) is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{f}(\tau)\, d\tau \tag{13.47}$$

Before proceeding further, we must define the exponential of the matrix appearing in Eq. (13.47). An exponential of a matrix is defined by an infinite series identical to that used in defining an exponential of a scalar. We shall define

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots + \frac{\mathbf{A}^nt^n}{n!} + \cdots + \tag{13.48a}$$
$$= \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!} \tag{13.48b}$$

Thus, if

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

then

$$\mathbf{A}t = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} t = \begin{bmatrix} 0 & t \\ 2t & t \end{bmatrix} \tag{13.49}$$

and

$$\frac{\mathbf{A}^2 t^2}{2!} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \frac{t^2}{2} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \frac{t^2}{2} = \begin{bmatrix} t^2 & \frac{t^2}{2} \\ t^2 & \frac{3t^2}{2} \end{bmatrix} \tag{13.50}$$

and so on.

We can show that the infinite series in Eq. (13.48) is absolutely and uniformly convergent for all values of t. Consequently, it can be differentiated or integrated term by term. Thus, to find $(d/dt)e^{\mathbf{A}t}$, we differentiate the series on the right-hand side of Eq. (13.48a) term by term:

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A} + \mathbf{A}^2 t + \frac{\mathbf{A}^3 t^2}{2!} + \frac{\mathbf{A}^4 t^3}{3!} + \cdots \tag{13.51a}$$

$$= \mathbf{A}\left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots\right]$$

$$= \mathbf{A}e^{\mathbf{A}t} \tag{13.51b}$$

Note that the infinite series on the right-hand side of Eq. (13.51a) also may be expressed as

$$\frac{d}{dt}e^{\mathbf{A}t} = \left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \cdots + \cdots\right]\mathbf{A}$$

$$= e^{\mathbf{A}t}\mathbf{A}$$

Hence

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A} \tag{13.52}$$

Also note that from the definition (13.48a), it follows that

$$e^{\mathbf{0}} = \mathbf{I} \tag{13.53a}$$

where

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we premultiply or postmultiply the infinite series for $e^{\mathbf{A}t}$ [Eq. (13.48a)] by an infinite series for $e^{-\mathbf{A}t}$, we find that

$$(e^{-\mathbf{A}t})(e^{\mathbf{A}t}) = (e^{\mathbf{A}t})(e^{-\mathbf{A}t}) = \mathbf{I} \tag{13.53b}$$

In Sec. B.6-3, we show that

$$\frac{d}{dt}(\mathbf{PQ}) = \frac{d\mathbf{P}}{dt}\mathbf{Q} + \mathbf{P}\frac{d\mathbf{Q}}{dt}$$

Using this relationship, we observe that

$$\begin{aligned}\frac{d}{dt}[e^{-\mathbf{A}t}\mathbf{x}] &= \left(\frac{d}{dt}e^{-\mathbf{A}t}\right)\mathbf{x} + e^{-\mathbf{A}t}\dot{\mathbf{x}} \\ &= -e^{-\mathbf{A}t}\mathbf{A}\mathbf{x} + e^{-\mathbf{A}t}\dot{\mathbf{x}} \qquad (13.54)\end{aligned}$$

We now premultiply both sides of Eq. (13.46) by $e^{-\mathbf{A}t}$ to yield

$$e^{-\mathbf{A}t}\dot{\mathbf{x}} = e^{-\mathbf{A}t}\mathbf{A}\mathbf{x} + e^{-\mathbf{A}t}\mathbf{B}\mathbf{f} \qquad (13.55a)$$

or

$$e^{-\mathbf{A}t}\dot{\mathbf{x}} - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x} = e^{-\mathbf{A}t}\mathbf{B}\mathbf{f} \qquad (13.55b)$$

A glance at Eq. (13.54) shows that the left-hand side of Eq. (13.55b) is $\frac{d}{dt}[e^{-\mathbf{A}t}]$. Hence

$$\frac{d}{dt}[e^{-\mathbf{A}t}] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{f}$$

The integration of both sides of this equation from 0 to t yields

$$e^{-\mathbf{A}t}\mathbf{x}\Big|_0^t = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{f}(\tau)\,d\tau \qquad (13.56a)$$

or

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{f}(\tau)\,d\tau \qquad (13.56b)$$

Hence

$$e^{-\mathbf{A}t}\mathbf{x} = \mathbf{x}(0) + \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{f}(\tau)\,d\tau \qquad (13.56c)$$

Premultiplying Eq. (13.56c) by $e^{\mathbf{A}t}$ and using Eq. (13.53b), we have

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}t}\mathbf{x}(0)}_{\text{zero-input component}} + \underbrace{\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{f}(\tau)\,d\tau}_{\text{zero-state component}} \qquad (13.57a)$$

This is the desired solution. The first term on the right-hand side represents $x(t)$ when the input $f(t) = 0$. Hence it is the zero-input component. The second term, by a similar argument, is seen to be the zero-state component.

The results of Eq. (13.57a) can be expressed more conveniently in terms of the matrix convolution. We can define the convolution of two matrices in a manner similar to the multiplication of two matrices, except that the multiplication of two elements is replaced by their convolution. For example,

$$\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} * \begin{bmatrix} g_1 & g_2 \\ g_3 & g_4 \end{bmatrix} = \begin{bmatrix} (f_1 * g_1 + f_2 * g_3) & (f_1 * g_2 + f_2 * g_4) \\ (f_3 * g_1 + f_4 * g_3) & (f_3 * g_2 + f_4 * g_4) \end{bmatrix}$$

Using this definition of matrix convolution, we can express Eq. (13.57a) as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} * \mathbf{Bf}(t) \tag{13.57b}$$

Note that the limits of the convolution integral [Eq. (13.57a)] are from 0 to t. Hence, all the elements of $e^{\mathbf{A}t}$ in the convolution term of Eq. (13.57b) are implicitly assumed to be multiplied by $u(t)$.

The result of Eq. (13.57) can be easily generalized for any initial value of t. It is left as an exercise for the reader to show that the solution of the state equation can be expressed as

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{Bf}(\tau)\,d\tau \tag{13.58}$$

Determining $e^{\mathbf{A}t}$

The exponential $e^{\mathbf{A}t}$ required in Eq. (13.57) can be computed from the definition in Eq. (13.51a). Unfortunately, this is an infinite series, and its computation can be quite laborious. Moreover, we may not be able to recognize the closed-form expression for the answer. There are several efficient methods of determining $e^{\mathbf{A}t}$ in closed form. It is shown in Sec. B.6-5 that for an $n \times n$ matrix $\mathbf{A}$,

$$e^{\mathbf{A}t} = \beta_0\mathbf{I} + \beta_1\mathbf{A} + \beta_2\mathbf{A}^2 + \cdots + \beta_{n-1}\mathbf{A}^{n-1} \tag{13.59a}$$

where

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \cdots \\ \beta_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \cdots \\ e^{\lambda_n t} \end{bmatrix}$$

and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n characteristic values (eigenvalues) of $\mathbf{A}$.

We can also determine $e^{\mathbf{A}t}$ by comparing Eqs. (13.57a) and (13.35b). It is clear that

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}[\mathbf{\Phi}(s)] \tag{13.59b}$$

$$= \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \tag{13.59c}$$

Thus, $e^{\mathbf{A}t}$ and $\mathbf{\Phi}(s)$ are a Laplace transform pair. To be consistent with Laplace transform notation, $e^{\mathbf{A}t}$ is often denoted by $\boldsymbol{\phi}(t)$, **the state transition matrix** (STM):

$$e^{\mathbf{A}t} = \boldsymbol{\phi}(t)$$

■ **Example 13.7**

Find the solution to the problem in Example 13.5 using the time-domain method. For this case, the characteristic roots are given by

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+12 & -\frac{2}{3} \\ 36 & s+1 \end{vmatrix} = s^2 + 13s + 36 = (s+4)(s+9) = 0$$

The roots are $\lambda_1 = -4$ and $\lambda_2 = -9$, so

$$\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = \begin{bmatrix}1 & -4\\ 1 & -9\end{bmatrix}^{-1}\begin{bmatrix}e^{-4t}\\ e^{-9t}\end{bmatrix} = \frac{1}{5}\begin{bmatrix}9e^{-4t} - 4e^{-9t}\\ e^{-4t} - e^{-9t}\end{bmatrix}$$

and

$$\begin{aligned} e^{\mathbf{A}t} &= \beta_0\mathbf{I} + \beta_1\mathbf{A} \\ &= \left(\frac{9}{5}e^{-4t} - \frac{4}{5}e^{-9t}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \left(\frac{1}{5}e^{-4t} - \frac{1}{5}e^{-9t}\right)\begin{bmatrix}-12 & \frac{2}{3}\\ -36 & -1\end{bmatrix} \\ &= \begin{bmatrix}\left(\frac{-3}{5}e^{-4t} + \frac{8}{5}e^{-9t}\right) & \frac{2}{15}(e^{-4t} - e^{-9t})\\ \frac{36}{5}(-e^{-4t} + e^{-9t}) & \left(\frac{8}{5}e^{-4t} - \frac{3}{5}e^{-9t}\right)\end{bmatrix} \end{aligned} \tag{13.60}$$

The zero-input component is given by [see Eq. (13.57a)]

$$\begin{aligned} e^{\mathbf{A}t}\mathbf{x}(0) &= \begin{bmatrix}\left(-\frac{3}{5}e^{-4t} + \frac{8}{5}e^{-9t}\right) & \frac{2}{15}(e^{-4t} - e^{-9t})\\ \frac{36}{5}(-e^{-4t} + e^{-9t}) & \left(\frac{8}{5}e^{-4t} - \frac{3}{5}e^{-9t}\right)\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix} \\ &= \begin{bmatrix}\left(\frac{-16}{15}e^{-4t} + \frac{46}{15}e^{-9t}\right)u(t)\\ \left(\frac{-64}{5}e^{-4t} + \frac{69}{5}e^{-9t}\right)u(t)\end{bmatrix} \end{aligned} \tag{13.61a}$$

Note the presence of $u(t)$ in Eq. (13.61a), indicating that the response begins at $t = 0$.

The zero-state component is $e^{\mathbf{A}t} * \mathbf{Bf}$ [see Eq. (13.57b)], where

$$\mathbf{Bf} = \begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}u(t) = \begin{bmatrix}\frac{1}{3}u(t)\\ u(t)\end{bmatrix}$$

and

$$e^{\mathbf{A}t} * \mathbf{Bf}(t) = \begin{bmatrix}(\frac{-3}{5}e^{-4t} + \frac{8}{5}e^{-9t})u(t) & \frac{2}{15}(e^{-4t} - e^{-9t})u(t)\\ \frac{36}{5}(-e^{-4t} + e^{-9t}u(t)) & (\frac{8}{5}e^{-4t} - \frac{3}{5}e^{-9t})u(t)\end{bmatrix} * \begin{bmatrix}\frac{1}{3}u(t)\\ u(t)\end{bmatrix}$$

Note again the presence of the term $u(t)$ in every element of $e^{\mathbf{A}t}$. This is the case because the limits of the convolution integral run from 0 to t [Eq. (13.56)]. Thus

$$\begin{aligned} e^{\mathbf{A}t} * \mathbf{Bf}(t) &= \begin{bmatrix}\left(-\frac{3}{5}e^{-4t} + \frac{8}{5}e^{-9t}\right)u(t) * \frac{1}{3}u(t) & \frac{2}{15}(e^{-4t} - e^{-9t})u(t) * u(t)\\ \frac{36}{5}(-e^{-4t} + e^{-9t})u(t) * \frac{1}{3}u(t) & \left(\frac{8}{5}e^{-4t} - \frac{3}{5}e^{-9t}\right)u(t) * u(t)\end{bmatrix} \\ &= \begin{bmatrix}-\frac{1}{15}e^{-4t}u(t) * u(t) + \frac{2}{5}e^{-9t}u(t) * u(t)\\ -\frac{4}{5}e^{-4t}u(t) * u(t) + \frac{9}{5}e^{-9t}u(t) * u(t)\end{bmatrix} \end{aligned}$$

Substitution for the above convolution integrals from the convolution table (Table 2.1) yields

$$e^{\mathbf{A}t} * \mathbf{B}\mathbf{f}(t) = \begin{bmatrix} -\frac{1}{60}(1-e^{-4t})u(t) + \frac{2}{45}(1-e^{-9t})u(t) \\ -\frac{1}{5}(1-e^{-4t})u(t) + \frac{1}{5}(1-e^{-9t})u(t) \end{bmatrix}$$

$$= \begin{bmatrix} (\frac{1}{36} + \frac{1}{60}e^{-4t} - \frac{2}{45}e^{-9t})u(t) \\ \frac{1}{5}(e^{-4t} - e^{-9t})u(t) \end{bmatrix} \tag{13.61b}$$

The sum of the two components [Eq. (13.61a) and Eq. (13.61b)] now gives the desired solution for $\mathbf{x}(t)$:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{36} - \frac{21}{20}e^{-4t} + \frac{136}{45}e^{-9t}\right)u(t) \\ \left(\frac{-63}{5}e^{-4t} + \frac{68}{5}e^{-9t}\right)u(t) \end{bmatrix} \tag{13.61c}$$

This result is consistent with the solution obtained by using the frequency-domain method [see Eq. (13.36b)]. Once the state variables x_1 and x_2 are found for $t \geq 0$, all the remaining variables are determined from the output equation. ■

The Output

The output equation is given by

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{f}(t)$$

The substitution of the solution for $\mathbf{x}$ [Eq. (13.57)] in this equation yields

$$\mathbf{y}(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t} * \mathbf{B}\mathbf{f}(t)] + \mathbf{D}\mathbf{f}(t) \tag{13.62a}$$

Since the elements of $\mathbf{B}$ are constants,

$$e^{\mathbf{A}t} * \mathbf{B}\mathbf{f}(t) = e^{\mathbf{A}t}\mathbf{B} * \mathbf{f}(t)$$

With this result, Eq. (13.62a) becomes

$$\mathbf{y}(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\mathbf{B} * \mathbf{f}(t)] + \mathbf{D}\mathbf{f}(t) \tag{13.62b}$$

Now recall that the convolution of $f(t)$ with the unit impulse $\delta(t)$ yields $f(t)$. Let us define a $j \times j$ diagonal matrix $\boldsymbol{\delta}(t)$ such that all its diagonal terms are unit impulse functions. It is then obvious that

$$\boldsymbol{\delta}(t) * \mathbf{f}(t) = \mathbf{f}(t)$$

and Eq. (13.62b) can be expressed as

$$\mathbf{y}(t) = \mathbf{C}[e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\mathbf{B} * \mathbf{f}(t)] + \mathbf{D}\boldsymbol{\delta}(t) * \mathbf{f}(t) \tag{13.63a}$$

$$= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + [\mathbf{C}e^{\mathbf{A}t}\mathbf{B} + \mathbf{D}\boldsymbol{\delta}(t)] * \mathbf{f}(t) \tag{13.63b}$$

With the notation $\boldsymbol{\phi}(t)$ for $e^{\mathbf{A}t}$, Eq. (13.63b) may be expressed as

$$\mathbf{y}(t) = \underbrace{\mathbf{C}\boldsymbol{\phi}(t)\mathbf{x}(0)}_{\text{zero-input response}} + \underbrace{[\mathbf{C}\boldsymbol{\phi}(t)\mathbf{B} + \mathbf{D}\boldsymbol{\delta}(t)] * \mathbf{f}(t)}_{\text{zero-state response}} \tag{13.63c}$$

The zero-state response; that is, the response when $\mathbf{x}(0) = \mathbf{0}$, is

$$\mathbf{y}(t) = [\mathbf{C}\boldsymbol{\phi}(t)\mathbf{B} + \mathbf{D}\boldsymbol{\delta}(t)] * \mathbf{f}(t) \tag{13.64a}$$

$$= \mathbf{h}(t) * \mathbf{f}(t) \tag{13.64b}$$

where

$$\mathbf{h}(t) = \mathbf{C}\boldsymbol{\phi}(t)\mathbf{B} + \mathbf{D}\boldsymbol{\delta}(t) \tag{13.65}$$

The matrix $\mathbf{h}(t)$ is a $k \times j$ matrix known as the **impulse response matrix**. The reason for this designation is obvious. The ijth element of $\mathbf{h}(t)$ is $h_{ij}(t)$, which represents the zero-state response y_i when the input $f_j(t) = \delta(t)$ and when all other inputs (and all the initial conditions) are zero. It can also be seen from Eq. (13.39) and (13.64b) that

$$\mathcal{L}[\mathbf{h}(t)] = \mathbf{H}(s)$$

■ Example 13.8

For the system described by Eqs. (13.40a) and (13.40b), determine $e^{\mathbf{A}t}$ using Eq. (13.59b):

$$\boldsymbol{\phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\boldsymbol{\Phi}(s)$$

This problem was solved earlier with frequency-domain techniques. From Eq. (13.41), we have

$$\boldsymbol{\phi}(t) = \mathcal{L}^{-1}\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

The same result is obtained in Sec. B.6-5 by using Eq. (13.59a) [see Eq. (B.84)].

Also, $\boldsymbol{\delta}(t)$ is a diagonal $j \times j$ or 2×2 matrix:

$$\boldsymbol{\delta}(t) = \begin{bmatrix} \delta(t) & 0 \\ 0 & \delta(t) \end{bmatrix}$$

Substituting the matrices $\boldsymbol{\phi}(t)$, $\boldsymbol{\delta}(t)$, $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{B}$ [Eq. (13.40c)] into Eq. (13.65), we have

$$\mathbf{h}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-+2} + e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta(t) & 0 \\ 0 & \delta(t) \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-t} - 2e^{-2t} & e^{-t} - e^{-2t} \\ \delta(t) + 2e^{-2t} & e^{-2t} \\ -6e^{-t} + 8e^{-2t} & \delta(t) - 2e^{-2t} + 4e^{-2t} \end{bmatrix} \tag{13.66}$$

The reader can verify that the transfer-function matrix $\mathbf{H}(s)$ in Eq. (13.42) is the Laplace transform of the unit-impulse response matrix $\mathbf{h}(t)$ in Eq. (13.66). ■

13.4 Linear Transformation of State Vectors

In Sec. 13.1 we saw that the state of a system can be specified in several ways. The sets of all possible state variables must be related—in other words, if we are given one set of state variables, we should be able to relate it to any other set. We are particularly interested in a linear type of relationship. Let $x_1, x_2, \ldots, x_n$ and $w_1, w_2, \ldots, w_n$ be two different sets of state variables specifying the same system. Let these sets be related by linear equations as

$$\begin{aligned} w_1 &= p_{11}x_1 + p_{12}x_2 + \cdots + p_{1n}x_n \\ w_2 &= p_{21}x_1 + p_{22}x_2 + \cdots + p_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ w_n &= p_{n1}x_1 + p_{n2}x_2 + \cdots + p_{nn}x_n \end{aligned} \tag{13.67a}$$

or

$$\underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \cdots \\ w_n \end{bmatrix}}_{\mathbf{w}} = \underbrace{\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}}_{\mathbf{x}} \tag{13.67b}$$

Defining the vector $\mathbf{w}$ and matrix $\mathbf{P}$, as shown above, we can write Eq. (13.67b) as

$$\mathbf{w} = \mathbf{P}\mathbf{x} \tag{13.67c}$$

and

$$\mathbf{x} = \mathbf{P}^{-1}\mathbf{w} \tag{13.67d}$$

Thus, the state vector $\mathbf{x}$ is transformed into another state vector $\mathbf{w}$ through the linear transformation in Eq. (13.67c).

If we know $\mathbf{w}$, we can determine $\mathbf{x}$ from Eq. (13.67d), provided that $\mathbf{P}^{-1}$ exists. This is equivalent to saying that $\mathbf{P}$ is a nonsingular matrix† ($|\mathbf{P}| \neq 0$). Thus, if $\mathbf{P}$ is a nonsingular matrix, the vector $\mathbf{w}$ defined by Eq. (13.67c) is also a state vector. Consider the state equation of a system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{f} \tag{13.68a}$$

If

$$\mathbf{w} = \mathbf{P}\mathbf{x} \tag{13.68b}$$

then

$$\mathbf{x} = \mathbf{P}^{-1}\mathbf{w}$$

†This condition is equivalent to saying that all n equations in Eq. (13.67a) are linearly independent; that is, none of the n equations can be expressed as a linear combination of the remaining equations.

and

$$\dot{\mathbf{x}} = \mathbf{P}^{-1}\dot{\mathbf{w}}$$

Hence the state equation (13.68a) now becomes

$$\mathbf{P}^{-1}\dot{\mathbf{w}} = \mathbf{A}\mathbf{P}^{-1}\mathbf{w} + \mathbf{B}\mathbf{f}$$

or

$$\dot{\mathbf{w}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{w} + \mathbf{P}\mathbf{B}\mathbf{f} \tag{13.68c}$$

$$= \hat{\mathbf{A}}\mathbf{w} + \hat{\mathbf{B}}\mathbf{f} \tag{13.68d}$$

where

$$\hat{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1} \tag{13.69a}$$

and

$$\hat{\mathbf{B}} = \mathbf{P}\mathbf{B} \tag{13.69b}$$

Equation (13.68d) is a state equation for the same system, but now it is expressed in terms of the state vector $\mathbf{w}$.

The output equation is also modified. Let the original output equation be

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{f}$$

In terms of the new state variable $\mathbf{w}$, this equation becomes

$$\mathbf{y} = \mathbf{C}(\mathbf{P}^{-1}\mathbf{w}) + \mathbf{D}\mathbf{f}$$

$$= \hat{\mathbf{C}}\mathbf{w} + \mathbf{D}\mathbf{f}$$

where

$$\hat{\mathbf{C}} = \mathbf{C}\mathbf{P}^{-1} \tag{13.69c}$$

■ Example 13.9

The state equations of a certain system are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} f(t) \tag{13.70a}$$

Find the state equations for this system when the new state variables w_1 and w_2 are

$$w_1 = x_1 + x_2$$

$$w_2 = x_1 - x_2$$

or

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{13.70b}$$

According to Eq. (13.70b), the state equation for the state variable $\mathbf{w}$ is given by

$$\dot{\mathbf{w}} = \hat{\mathbf{A}}\mathbf{w} + \hat{\mathbf{B}}\mathbf{f}$$

where [see Eq. (13.69)]

$$\hat{\mathbf{A}} = \mathbf{PAP}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

and

$$\hat{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} f(t)$$

This is the desired state equation for the state vector $\mathbf{w}$. The solution of this equation requires a knowledge of the initial state $\mathbf{w}(0)$. This can be obtained from the given initial state $\mathbf{x}(0)$ by using Eq. (13.70b). ■

⊙ **Computer Example C13.4**
Solve Example 13.9 using MATLAB.

```
A=[0 1;-2 -3]; B=[1; 2];
P=[1 1;1 -1];
Ahat=P*A*inv(P)
Bhat=P*B
```
⊙

Invariance of Eigenvalues

We have seen (Sec. 13.3) that the poles of all possible transfer functions of a system are the eigenvalues of the matrix $\mathbf{A}$. If we transform a state vector from $\mathbf{x}$ to $\mathbf{w}$, the variables $w_1, w_2, \ldots, w_n$ are linear combinations of $x_1, x_2, \ldots, x_n$ and therefore may be considered as outputs. Hence, the poles of the transfer functions relating $w_1, w_2, \ldots, w_n$ to the various inputs must also be the eigenvalues of matrix $\mathbf{A}$. On the other hand, the system is also specified by Eq. (13.68d). This means that the poles of the transfer functions must be the eigenvalues of $\hat{\mathbf{A}}$. Therefore, the eigenvalues of matrix $\mathbf{A}$ remain unchanged for the linear transformation of variables represented by Eq. (13.67), and the eigenvalues of matrix $\mathbf{A}$ and matrix $\hat{\mathbf{A}}(\hat{\mathbf{A}} = \mathbf{PAP}^{-1})$ are identical, implying that the characteristic equations of $\mathbf{A}$ and $\hat{\mathbf{A}}$ are also identical. This result also can be proved alternately as follows.

Consider the matrix $\mathbf{P}(s\mathbf{I} - \mathbf{A})\mathbf{P}^{-1}$. We have

$$\mathbf{P}(s\mathbf{I} - \mathbf{A})\mathbf{P}^{-1} = \mathbf{P}s\mathbf{I}\mathbf{P}^{-1} - \mathbf{PAP}^{-1} = s\mathbf{PIP}^{-1} - \hat{\mathbf{A}} = s\mathbf{I} - \hat{\mathbf{A}}$$

Taking the determinants of both sides, we obtain

$$|\mathbf{P}||s\mathbf{I}-\mathbf{A}||\mathbf{P}^{-1}| = |s\mathbf{I}-\hat{\mathbf{A}}|$$

The determinants $|\mathbf{P}|$ and $|\mathbf{P}^{-1}|$ are reciprocals of each other. Hence

$$|s\mathbf{I}-\mathbf{A}| = |s\mathbf{I}-\hat{\mathbf{A}}| \tag{13.71}$$

This is the desired result. We have shown that the characteristic equations of $\mathbf{A}$ and $\hat{\mathbf{A}}$ are identical. Hence the eigenvalues of $\mathbf{A}$ and $\hat{\mathbf{A}}$ are identical.

In Example 13.9, matrix $\mathbf{A}$ is given as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The characteristic equation is

$$|s\mathbf{I}-\mathbf{A}| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s^2+3s+2 = 0$$

Also

$$\hat{\mathbf{A}} = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

and

$$|s\mathbf{I}-\hat{\mathbf{A}}| = \begin{bmatrix} s+2 & 0 \\ -3 & s+1 \end{bmatrix} = s^2+3s+2 = 0$$

This result verifies that the characteristic equations of $\mathbf{A}$ and $\hat{\mathbf{A}}$ are identical.

13.4-1 Diagonalization of Matrix A

For several reasons, it is desirable to make matrix $\mathbf{A}$ diagonal. If $\mathbf{A}$ is not diagonal, we can transform the state variables such that the resulting matrix $\hat{\mathbf{A}}$ is diagonal.† One can show that for any diagonal matrix $\mathbf{A}$, the diagonal elements of this matrix must necessarily be $\lambda_1, \lambda_2, \ldots, \lambda_n$ (the eigenvalues) of the matrix. Consider the diagonal matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & a_n \end{bmatrix}$$

The characteristic equation is given by

†In this discussion we assume distinct eigenvalues. If the eigenvalues are not distinct, we can reduce the matrix to a modified diagonalized (Jordan) form.

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} (s-a_1) & 0 & 0 & \cdots & 0 \\ 0 & (s-a_2) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & (s-a_n) \end{vmatrix} = 0$$

or

$$(s-a_1)(s-a_2)\cdots(s-a_n) = 0$$

Hence, the eigenvalues of $\mathbf{A}$ are $a_1, a_2, \ldots, a_n$. The nonzero (diagonal) elements of a diagonal matrix are therefore its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We shall denote the diagonal matrix by a special symbol, Λ:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{13.72}$$

Let us now consider the transformation of the state vector A such that the resulting matrix $\hat{\mathbf{A}}$ is a diagonal matrix Λ.

Consider the system

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bf}$$

We shall assume that $\lambda_1, \lambda_2, \ldots, \lambda_n$, the eigenvalues of $\mathbf{A}$, are distinct (no repeated roots). Let us transform the state vector $\mathbf{x}$ into the new state vector $\mathbf{z}$, using the transformation

$$\mathbf{z} = \mathbf{Px} \tag{13.73a}$$

Then, after the development of Eq. (13.68c), we have

$$\dot{\mathbf{z}} = \mathbf{PAP}^{-1}\mathbf{z} + \mathbf{PBf} \tag{13.73b}$$

We desire the transformation to be such that $\mathbf{PAP}^{-1}$ is a diagonal matrix Λ given by Eq. (13.72), or

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \hat{\mathbf{B}}\mathbf{f} \tag{13.73c}$$

Hence

$$\Lambda = \mathbf{PAP}^{-1} \tag{13.74a}$$

or

$$\Lambda\mathbf{P} = \mathbf{PA} \tag{13.74b}$$

We know Λ and $\mathbf{A}$. Equation (13.74b) therefore can be solved to determine $\mathbf{P}$.

■ **Example 13.10**

Find the diagonalized form of the state equation for the system in Example 13.9.

In this case,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

We found $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

and Eq. (13.74b) becomes

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Equating the four elements on two sides, we obtain

$$-p_{11} = -2p_{12} \tag{13.75a}$$

$$-p_{12} = p_{11} - 3p_{12} \tag{13.75b}$$

$$-2p_{21} = -2p_{22} \tag{13.75c}$$

$$-2p_{22} = p_{21} - 3p_{22} \tag{13.75d}$$

The reader will immediately recognize that Eqs. (13.75a) and (13.75b) are identical. Similarly, Eqs. (13.75c) and (13.75d) are identical. Hence two equations may be discarded, leaving us with only two equations [Eqs. (13.75a) and (13.75c)] and four unknowns. This observation means there is no unique solution. There is, in fact, an infinite number of solutions. We can assign any value to p_{11} and p_{21} to yield one possible solution.† If $p_{11} = k_1$ and $p_{21} = k_2$, then from Eqs. (13.75a) and (13.75c) we have $p_{12} = k_1/2$ and $p_{22} = k_2$:

$$\mathbf{P} = \begin{bmatrix} k_1 & \frac{k_1}{2} \\ k_2 & k_2 \end{bmatrix} \tag{13.75e}$$

We may assign any values to k_1 and k_2. For convenience, let $k_1 = 2$ and $k_2 = 1$. This substitution yields

$$\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \tag{13.75f}$$

The transformed variables [Eq. (13.73a)] are

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} \tag{13.76}$$

Thus, the new state variables z_1 and z_2 are related to x_1 and x_2 by Eq. (13.76). The system equation with $\mathbf{z}$ as the state vector is given by [see Eq. (13.73c)]

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \hat{\mathbf{B}}\mathbf{f}$$

†If, however, we want the state equations in diagonalized form, as in Eq. (13.29a), where all the elements of $\hat{\mathbf{B}}$ matrix are unity, there is a unique solution. The reason is that the equation $\hat{\mathbf{B}} = \mathbf{PB}$, where all the elements of $\hat{\mathbf{B}}$ are unity, imposes additional constraints. In the present example, this condition will yield $p_{11} = \frac{1}{2}$, $p_{12} = \frac{1}{4}$, $p_{21} = \frac{1}{3}$, and $p_{22} = \frac{1}{3}$. The relationship between $\mathbf{z}$ and $\mathbf{x}$ is then

$$z_1 = \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \quad \text{and} \quad z_2 = \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2$$

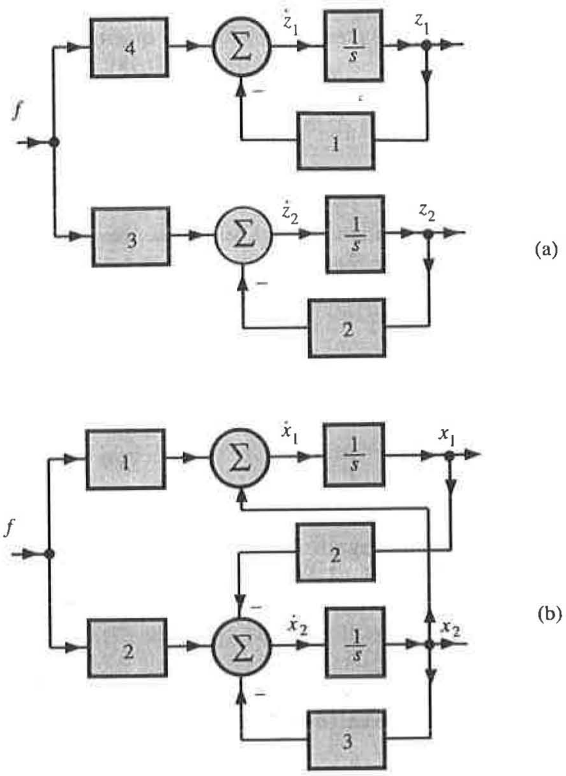


Fig. 13.8 Two realizations of the second-order system in Example 13.10.

where

$$\hat{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Hence

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} f \tag{13.77a}$$

or

$$\dot{z}_1 = -z_1 + 4f$$

$$\dot{z}_2 = -2z_2 + 3f \tag{13.77b}$$

Note the distinctive nature of these state equations. Each state equation involves only one variable and therefore can be solved by itself. A general state equation has the derivative of one state variable equal to a linear combination of all state variables. Such is not the case with the diagonalized matrix Λ. Each state variable z_i is chosen so that it is uncoupled from the rest of the variables; hence a system with n eigenvalues is split into n decoupled systems, each with an equation of the form

$$\dot{z}_i = \lambda_i z_i + \text{(input terms)}$$

This fact also can be readily seen from Fig. 13.8a, which is a realization of the system represented by Eq. (13.77). In contrast, consider the original state equations [see Eq. 13.70a)]

$$\dot{x}_1 = x_2 + f(t)$$

$$\dot{x}_2 = -2x_1 - 3x_2 + 2f(t)$$

A realization for these equations is shown in Fig. 13.8b. It can be seen from Fig. 13.8a that the states z_1 and z_2 are decoupled, whereas the states x_1 and x_2 (Fig. 13.8b) are coupled. It should be remembered that Figs. 13.8a and 13.8b are simulations of the same system.† ■

⊙ **Computer Example C13.5**
Solve Example 13.10 using MATLAB.
Caution: The answer for $\hat{\mathbf{B}}$ is not unique.

```
A=[0 1;-2 -3]; B=[1; 2];
[V, L]=eig(A);
P=inv(V);
Lambda=P*A*inv(P);
Bhat=P*B
```
⊙

13.5 Controllability and Observability

Consider a diagonalized state-space description of a system

$$\dot{\mathbf{z}} = \Lambda \mathbf{z} + \hat{\mathbf{B}}\mathbf{f} \tag{13.78a}$$

and

$$\mathbf{Y} = \hat{\mathbf{C}}\mathbf{z} + \mathbf{D}\mathbf{f} \tag{13.78b}$$

We shall assume that all n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. The state equations (13.78a) are of the form

$$\dot{z}_m = \lambda_m z_m + \hat{b}_{m1} f_1 + \hat{b}_{m2} f_2 + \cdots + \hat{b}_{mj} f_j \qquad m = 1, 2, \ldots n$$

If $\hat{b}_{m1}, \hat{b}_{m2}, \ldots, \hat{b}_{mj}$ (the mth row in matrix $\hat{\mathbf{B}}$) are all zero, then

$$\dot{z}_m = \lambda_m z_m$$

and the variable z_m is uncontrollable because z_m is not connected to any of the inputs. Moreover, z_m is decoupled from all the remaining $(n-1)$ state variables because of the diagonalized nature of the variables. Hence, there is no direct or indirect coupling of z_m with any of the inputs, and the system is uncontrollable. In contrast, if at least one element in the mth row of $\hat{\mathbf{B}}$ is nonzero, z_m is coupled to at least one input and is therefore controllable. *Thus, a system with a diagonalized*

†Here we only have a simulated state equation; the outputs are not shown. The outputs are linear combinations of state variables (and inputs). Hence, the output equation can be easily incorporated into these diagrams (see Fig. 13.7).

state [Eqs. (13.78)] is completely controllable if and only if the matrix $\hat{\mathbf{B}}$ *has no row of zero elements.*

The outputs [Eq.(13.78b)] are of the form

$$y_i = \hat{c}_{i1}z_1 + \hat{c}_{i2}z_2 + \cdots + \hat{c}_{in}z_n + \sum_{m=1}^{j} d_{im}f_m$$

If $\hat{c}_{im} = 0$, then the state z_m will not appear in the expression for y_i. Since all the states are decoupled because of the diagonalized nature of the equations, the state z_m cannot be observed directly or indirectly (through other states) at the output y_i. Hence the mth mode $e^{\lambda_m t}$ will not be observed at the output y_i. If $\hat{c}_{1m}, \hat{c}_{2m}, \ldots, \hat{c}_{km}$ (the mth column in matrix $\hat{\mathbf{C}}$) are all zero, the state z_m will not be observable at any of the k outputs, and the state z_m is unobservable. In contrast, if at least one element in the mth column of $\hat{\mathbf{C}}$ is nonzero, z_m is observable at least at one output. *Thus, a system with diagonalized equations of the form in Eqs. (13.78) is completely observable if and only if the matrix* $\hat{\mathbf{C}}$ *has no column of zero elements.* In the above discussion, we assumed distinct eigenvalues; for repeated eigenvalues, the modified criteria can be found in the literature.[1,2]

If the state-space description is not in diagonalized form, it may be converted into diagonalized form using the procedure in Example 13.10. It is also possible to test for controllability and observability even if the state-space description is in undiagonalized form.[1,2]†

■ Example 13.11

Investigate the controllability and observability of the systems in Figs. 13.9a and 13.9b.

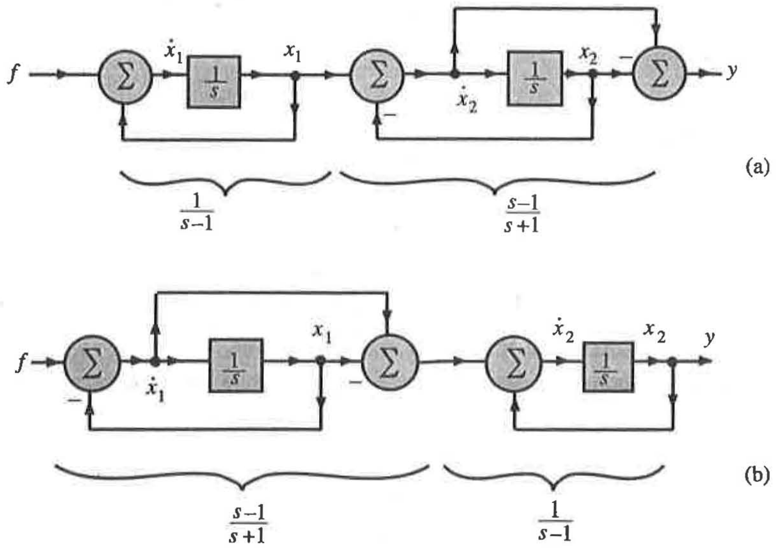


Fig. 13.9 Systems for Example 13.11.

---

†We can show that a system is completely controllable if and only if the $n \times nj$ composite matrix $[\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \ldots, \mathbf{A}^{n-1}\mathbf{B}]$ has a rank n. Similarly, a system is completely observable if and only if the $n \times nk$ composite matrix $[\mathbf{C}', \mathbf{A}'\mathbf{C}', \mathbf{A}'^2\mathbf{C}', \ldots, \mathbf{A}'^{n-1}\mathbf{C}']$ has a rank n.

In both cases, the state variables are identified as the two integrator outputs, x_1 and x_2. The state equations for the system in Fig. 13.9a are

$$\dot{x}_1 = x_1 + f$$
$$\dot{x}_2 = x_1 - x_2 \tag{13.79}$$

and

$$y = x_1 - 2x_2$$

Hence

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [1 \quad -2], \quad \mathbf{D} = 0$$

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s-1 & 0 \\ -1 & s+1 \end{vmatrix} = (s-1)(s+1)$$

Therefore

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -1$$

and

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{13.80}$$

We shall now use the procedure in Sec. 13.4-1 to diagonalize this system. According to Eq. (13.74b), we have

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

The solution of this equation yields

$$p_{12} = 0 \quad \text{and} \quad -2p_{21} = p_{22}$$

Choosing $p_{11} = 1$ and $p_{21} = 1$, we have

$$p = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$$

and

$$\hat{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{13.81a}$$

All the rows of $\hat{\mathbf{B}}$ are nonzero. Hence the system is controllable. Also,

$$\begin{aligned} \mathbf{Y} &= \mathbf{Cx} \\ &= \mathbf{CP}^{-1}\mathbf{z} \\ &= \hat{\mathbf{C}}\mathbf{z} \end{aligned} \tag{13.81b}$$

and

$$\hat{\mathbf{C}} = \mathbf{CP}^{-1} = [1 \quad -2] \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}^{-1} = [1 \quad -2] \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = [0 \quad 1] \tag{13.81c}$$

The first column of $\hat{\mathbf{C}}$ is zero. Hence the mode z_1 (corresponding to $\lambda_1 = 1$) is unobservable. The system is therefore controllable but not observable. We come to the same conclusion by realizing the system with the state variables z_1 and z_2, whose state equations are

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \hat{\mathbf{B}}f$$

$$y = \hat{\mathbf{C}}\mathbf{z}$$

According to Eqs. (13.80) and (13.81), we have

$$\dot{z}_1 = z_1 + f$$

$$\dot{z}_2 = -z_2 + f$$

and

$$y = z_2$$

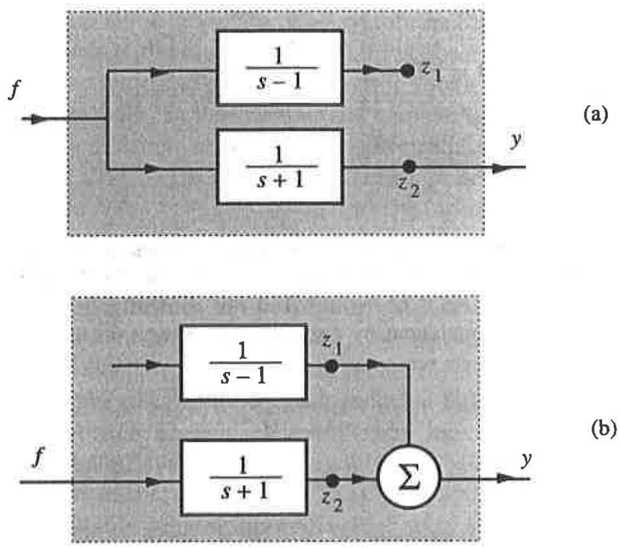


Fig. 13.10 Equivalent of the systems in Fig. 13.9.

Figure 13.10a shows a realization of these equations. It is clear that each of the two modes is controllable, but the first mode (corresponding to $\lambda = 1$) is not observable at the output.

The state equations for the system in Fig. 13.9b are

$$\dot{x}_1 = -x_1 + f$$

$$\dot{x}_2 = -2x_1 + x_2 + f \tag{13.82}$$

and

$$y = x_2$$

Hence

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [0 \quad 1], \quad \mathbf{D} = 0$$

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+1 & 0 \\ -1 & s-1 \end{vmatrix} = (s+1)(s-1)$$

so that $\lambda_1 = -1$, $\lambda_2 = 1$, and

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13.83}$$

Diagonalizing the matrix, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$$

The solution of this equation yields $p_{11} = -p_{12}$ and $p_{22} = 0$. Choosing $p_{11} = -1$ and $p_{21} = 1$, we obtain

$$\mathbf{P} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\hat{\mathbf{B}} = \mathbf{PB} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{13.84a}$$

$$\hat{\mathbf{C}} = \mathbf{CP}^{-1} = [0 \quad 1] \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = [1 \quad 1] \tag{13.84b}$$

The first row of $\hat{\mathbf{B}}$ is zero. Hence the second mode (corresponding to $\lambda_1 = 1$) is not controllable. However, since none of the columns of $\hat{\mathbf{C}}$ vanish, both modes are observable at the output. Hence the system is observable but not controllable.

We reach to the same conclusion by realizing the system with the state variables z_1 and z_2. The two state equations are

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + \hat{\mathbf{B}}f$$

$$y = \hat{\mathbf{C}}\mathbf{z}$$

From Eqs. (13.83) and (13.84), we have

$$\dot{z}_1 = z_1$$

$$\dot{z}_2 = -z_2 + f$$

and thus

$$y = z_1 + z_2 \tag{13.85}$$

Figure 13.10b shows a realization of these equations. Clearly, each of the two modes is observable at the output, but the mode corresponding to $\lambda_1 = 1$ is not controllable. ■

Computer Example C13.6

Solve Example 13.11 using MATLAB.

```
A=[1 0;1 -1]; B=[1; 0]; C=[1 -2];
[V, L]=eig(A);
P=inv(V);
```

```
Ahat=P*A*inv(P);
Bhat=P*B
Chat=C*inv(P)
```
⊙

13.5-1 Inadequacy of the Transfer Function Description of a System

Example 13.11 demonstrates the inadequacy of the transfer function to describe an LTI system in general. The systems in Figs. 13.9a and 13.9b both have the same transfer function

$$H(s) = \frac{1}{s+1}$$

Yet the two systems are very different. Their true nature is revealed in Figs. 13.10a and 13.10b, respectively. Both the systems are unstable, but their transfer function $H(s) = \frac{1}{s+1}$ does not give any hint of it. The system in Fig. 13.9a appears stable from the external terminals, but it is internally unstable. The system in Fig. 13.9b, on the other hand, will show instability at the external terminals, but its transfer function $H(s) = \frac{1}{s+1}$ is silent about it. The system in Fig. 13.9a is controllable but not observable, whereas the system in Fig. 13.9b is observable but not controllable.

The transfer function description of a system looks at a system only from the input and output terminals. Consequently, the transfer description can specify only the part of the system which is coupled to the input and the output terminals. Figures 13.10a and 13.10b show that in both cases only a part of the system that has a transfer function $H(s) = \frac{1}{s+1}$ is coupled to the input and the output terminals. This is the reason why both systems have the same transfer function $H(s) = \frac{1}{s+1}$.

The state variable description (Eqs. 13.79 and 13.82), on the other hand, contains all the information about these systems to describe them completely. The reason is that the state variable description is an internal description, not the external description obtained from the system behavior at external terminals.

Mathematically, the reason the transfer function fails to describe these systems completely is the fact that their transfer function has a common factor $s - 1$ in the numerator and denominator; this common factor is canceled out with a consequent loss of the information about these systems. Such a situation occurs when a system is uncontrollable and/or unobservable. If a system is both controllable and observable (which is the case with most of the practical systems) the transfer function describes the system completely. In such a case the internal and external descriptions are equivalent.

13.6 State-Space Analysis of Discrete-Time Systems

We have shown that an nth-order differential equation can be expressed in terms of n first-order differential equations. In the following analogous procedure, we show that an nth-order difference equation can be expressed in terms of n first-order difference equations.

Consider the z-transfer function

$$H[z] = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} \tag{13.86a}$$

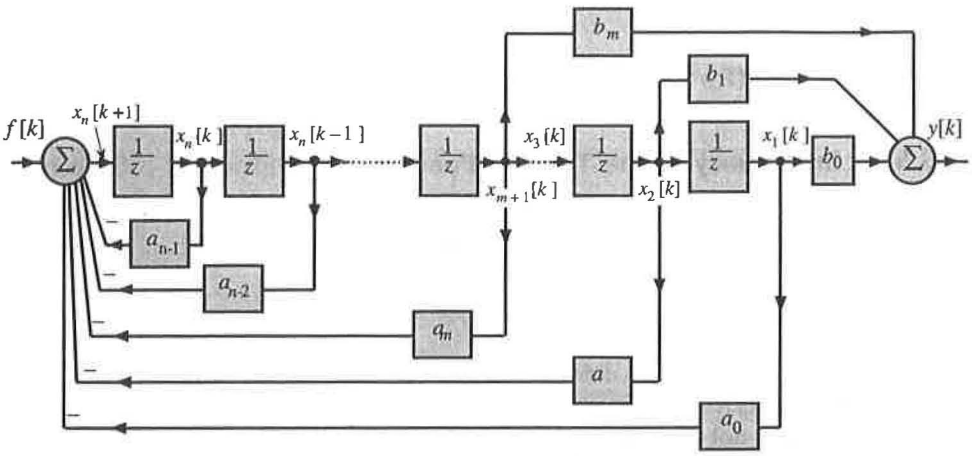


Fig. 13.11 Controller canonical realization of an nth-order discrete-time system.

The input $f[k]$ and the output $y[k]$ of this system are related by the difference equation

$$(E^n + a_{n-1}E^{n-1} + \cdots + a_1E + a_0)y[k] = (b_mE^m + b_{m-1}E^{m-1} + \cdots + b_1E + b_0)f[k] \tag{13.86b}$$

The controller canonical realization of this equation is illustrated in Fig. 13.11. Signals appearing at the outputs of n delay elements are denoted by $x_1[k]$, $x_2[k]$, ..., $x_n[k]$. The input of the first delay is $x_n[k+1]$. We can now write n equations, one at the input of each delay:

$$\begin{aligned} x_1[k+1] &= x_2[k] \\ x_2[k+1] &= x_3[k] \\ &\cdots\cdots\cdots \\ x_{n-1}[k+1] &= x_n[k] \\ x_n[k+1] &= -a_0x_1[k] - a_1x_2[k] - \cdots - a_{n-1}x_n[k] + f[k] \end{aligned} \tag{13.87}$$

and

$$y(k) = b_0x_1(k) + b_1x_2(k) + \cdots + b_mx_{m+1}(k) \tag{13.88}$$

Equations (13.87) are n first-order difference equations in n variables $x_1(k)$, $x_2(k)$, ..., $x_n(k)$. These variables should immediately be recognized as state variables, since the specification of the initial values of these variables in Fig. 13.11 will uniquely determine the response $y[k]$ for a given $f[k]$. Thus, Eqs. (13.87) represent the state equations, and Eq. (13.88) is the output equation. In matrix form we can write these equations as

$$\underbrace{\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ \cdots \\ x_{n-1}[k+1] \\ x_n[k+1] \end{bmatrix}}_{\mathbf{x}[k+1]} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1[k] \\ x_2[k] \\ \cdots \\ x_{n-1}[k] \\ x_n[k] \end{bmatrix}}_{\mathbf{x}[k]} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} f[k] \tag{13.89a}$$

and

$$\mathbf{y}[k] = \underbrace{\begin{bmatrix} b_0 & b_1 & \cdots & b_m \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1[k] \\ x_2[k] \\ \vdots \\ x_{m+1}[k] \end{bmatrix} \tag{13.89b}$$

In general,

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{f}[k] \tag{13.90a}$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{f}[k] \tag{13.90b}$$

Here we have represented a discrete-time system with state equations in controller canonical form. There are several other possible representations, as discussed in Sec. 13.2. We may, for example, realize the system by using a series, parallel, or observer canonical form. In all cases, the output of each delay element qualifies as a state variable. We then write the equation at the input of each delay element. The n equations thus obtained are the n state equations.

13.6-1 Solution in State-Space

Consider the state equation

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{f}[k] \tag{13.91}$$

From this equation it follows that

$$\mathbf{x}[k] = \mathbf{A}\mathbf{x}[k-1] + \mathbf{B}\mathbf{f}[k-1] \tag{13.92a}$$

and

$$\mathbf{x}[k-1] = \mathbf{A}\mathbf{x}[k-2] + \mathbf{B}\mathbf{f}[k-2] \tag{13.92b}$$

$$\mathbf{x}[k-2] = \mathbf{A}\mathbf{x}[k-3] + \mathbf{B}\mathbf{f}[k-3] \tag{13.92c}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbf{x}[1] = \mathbf{A}\mathbf{x}[0] + \mathbf{B}\mathbf{f}[0]$$

Substituting Eq. (13.92b) in Eq. (13.92a), we obtain

$$\mathbf{x}[k] = \mathbf{A}^2\mathbf{x}[k-2] + \mathbf{ABf}[k-2] + \mathbf{Bf}[k-1]$$

Substituting Eq. (13.92c) in this equation, we obtain

$$\mathbf{x}[k] = \mathbf{A}^3\mathbf{x}[k-3] + \mathbf{A}^2\mathbf{Bf}[k-3] + \mathbf{ABf}[k-2] + \mathbf{Bf}[k-1]$$

Continuing in this way, we obtain

$$\begin{aligned}\mathbf{x}[k] &= \mathbf{A}^k\mathbf{x}[0] + \mathbf{A}^{k-1}\mathbf{Bf}[0] + \mathbf{A}^{k-2}\mathbf{Bf}[1] + \cdots + \mathbf{Bf}[k-1] \\ &= \mathbf{A}^k\mathbf{x}[0] + \sum_{j=0}^{k-1}\mathbf{A}^{k-1-j}\mathbf{Bf}[j] \qquad (13.93a)\end{aligned}$$

The upper limit on the summation in Eq. (13.93a) is nonnegative. Hence $k \geq 1$, and the summation is recognized as the convolution sum

$$\mathbf{A}^{k-1}u[k-1] * \mathbf{Bf}[k]$$

Hence

$$\mathbf{x}[k] = \underbrace{\mathbf{A}^k\mathbf{x}[0]}_{\text{zero-input}} + \underbrace{\mathbf{A}^{k-1}u[k-1] * \mathbf{Bf}[k]}_{\text{zero-state}} \qquad (13.93b)$$

and

$$\begin{aligned}\mathbf{y}[k] &= \mathbf{Cx} + \mathbf{Df} \\ &= \mathbf{CA}^k\mathbf{x}[0] + \sum_{j=0}^{k-1}\mathbf{CA}^{k-1-j}\mathbf{Bf}[j] + \mathbf{Df} \qquad (13.94a) \\ &= \mathbf{CA}^k\mathbf{x}[0] + \mathbf{CA}^{k-1}u[k-1] * \mathbf{Bf}[k] + \mathbf{Df} \qquad (13.94b)\end{aligned}$$

In Sec. B.6-5, we showed that

$$\mathbf{A}^k = \beta_0\mathbf{I} + \beta_1\mathbf{A} + \beta_2\mathbf{A}^2 + \cdots + \beta_{n-1}\mathbf{A}^{n-1} \qquad (13.95a)$$

where (assuming n distinct eigenvalues of $\mathbf{A}$)

$$\begin{bmatrix}\beta_0 \\ \beta_1 \\ \cdots \\ \beta_{n-1}\end{bmatrix} = \begin{bmatrix}1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1}\end{bmatrix}^{-1}\begin{bmatrix}\lambda_1^k \\ \lambda_2^k \\ \cdots \\ \lambda_n^k\end{bmatrix} \qquad (13.95b)$$

and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n eigenvalues of $\mathbf{A}$.

We can also determine $\mathbf{A}^k$ from the z-transform formula, which will be derived later in Eq. (13.102):

$$\mathbf{A}^k = \mathcal{Z}^{-1}[(\mathbf{I} - z^{-1}\mathbf{A})^{-1}] \qquad (13.95c)$$

■ **Example 13.12**

Give a state-space description of the system in Fig. 13.12. Find the output $y[k]$ if the input $f[k] = u[k]$ and the initial conditions are $x_1[0] = 2$ and $x_2[0] = 3$.

The state equations are [see Eq. (13.89)]

$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \tag{13.96a}$$

and

$$y[k] = [-1 \quad 5] \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} \tag{13.96b}$$

To find the solution [Eq. (13.94)], we must first determine $\mathbf{A}^k$. The characteristic equation of $\mathbf{A}$ is

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 \\ \frac{1}{6} & \lambda - \frac{5}{6} \end{vmatrix} = \lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = \left(\lambda - \frac{1}{3}\right)\left(\lambda - \frac{1}{2}\right) = 0$$

Hence, $\lambda_1 = \frac{1}{3}$ and $\lambda_2 = \frac{1}{2}$ are the eigenvalues of $\mathbf{A}$ and [see Eq. (13.95)]

$$\mathbf{A}^k = \beta_0\mathbf{I} + \beta_1\mathbf{A}$$

where [see Eq. (B.95b)]

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} \\ 1 & \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} (\frac{1}{3})^k \\ (\frac{1}{2})^k \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} (3)^{-k} \\ (2)^{-k} \end{bmatrix} = \begin{bmatrix} 3(3)^{-k} - 2(2)^{-k} \\ -6(3)^{-k} + 6(2)^{-k} \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{A}^k &= [3(3)^{-k} - 2(2)^{-k}] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + [-6(3)^{-k} + 6(2)^{-k}] \begin{bmatrix} 0 & 1 \\ -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \\ &= \begin{bmatrix} 3(3)^{-k} - 2(2)^{-k} & -6(3)^{-k} + 6(2)^{-k} \\ (3)^{-k} - (2)^{-k} & -2(3)^{-k} + 3(2)^{-k} \end{bmatrix} \end{aligned} \tag{13.97}$$

We can now determine the state vector $\mathbf{x}[k]$ from Eq. (13.93b). Since we are interested in the output $y[k]$, we shall use Eq. (13.94b) directly. Note that

$$\mathbf{CA}^k = [-1 \quad 5]\,\mathbf{A}^k = [2(3)^{-k} - 3(2)^{-k} \quad -4(3)^{-k} + 9(2)^{-k}] \tag{13.98}$$

and the zero-input response is $\mathbf{CA}^k\mathbf{x}[0]$, with

$$\mathbf{x}[0] = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Hence, the zero-input response is

$$\mathbf{CA}^k\mathbf{x}[0] = -8(3)^{-k} + 21(2)^{-k} \tag{13.99a}$$

The zero-state component is given by the convolution sum of $\mathbf{CA}^{k-1}u[k-1]$ and $\mathbf{Bf}[k]$. Using the shifting property of the convolution sum [Eq. (9.46)], we can obtain the zero-state component by finding the convolution sum of $\mathbf{CA}^k u[k]$ and $\mathbf{Bf}[k]$ and then replacing

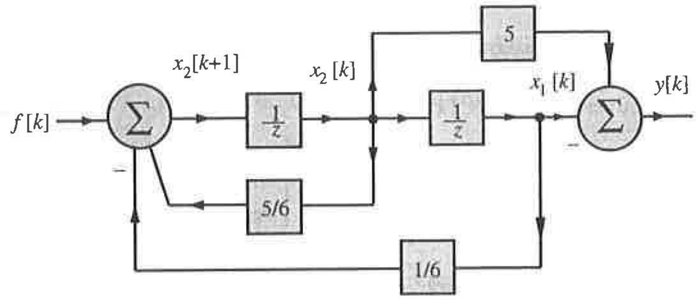


Fig. 13.12 System for Example 13.12.

k with $k-1$ in the result. We use this procedure because the convolution sums are listed in Table 9.1 for functions of the type $f[k]u[k]$ rather than $f[k]u[k-1]$.

$$\mathbf{C}\mathbf{A}^k u[k] * \mathbf{B}f[k] = [\,2(3)^{-k} - 3(2)^{-k} \quad -4(3)^{-k} + 9(2)^{-k}\,] * \begin{bmatrix} 0 \\ u[k] \end{bmatrix}$$

$$= -4(3)^{-k} * u[k] + 9(2)^{-k} * u[k]$$

Using Table 9.1 (Pair 2a), we obtain

$$\mathbf{C}\mathbf{A}^k u[k] * \mathbf{B}f[k] = -4\left[\frac{1-3^{-(k+1)}}{1-\frac{1}{3}}\right]u[k] + 9\left[\frac{1-2^{-(k+1)}}{1-\frac{1}{2}}\right]u[k]$$

$$= [12 + 6(3^{-(k+1)}) - 18(2^{-(k+1)})]u[k]$$

Now the desired (zero-state) response is obtained by replacing k by $k-1$. Hence

$$\mathbf{C}\mathbf{A}^k u[k] * \mathbf{B}f[k-1] = [12 + 6(3)^{-k} - 18(2)^{-k}]u[k-1] \tag{13.99b}$$

It follows that

$$y[k] = [-8(3)^{-k} + 21(2)^{-k}u[k] + [12 + 6(3)^{-k} - 18(2)^{-k}]u[k-1] \tag{13.100a}$$

This is the desired answer. We can simplify this answer by observing that $12 + 6(3)^{-k} - 18(2)^{-k} = 0$ for $k = 0$. Hence, $u[k-1]$ may be replaced by $u[k]$ in Eq. (13.99b), and

$$y[k] = [12 - 2(3)^{-k} + 3(2)^{-k}]u[k] \tag{13.100b}$$

■

⊙ **Computer Example C13.7**

Solve Example 13.12 using MATLAB.

```
A=[0 1;-1/6 5/6]; B=[0; 1]; C=[-1 5]; D=0;
x0=[2;3];
k=0:25;
u=ones(1,26);
[y,x]=dlsim(A,B,C,D,u,x0);
stem(k,y)
```
⊙

⊙ **Computer Example C13.8**
Using MATLAB find the zero-state response of the system in Example 13.12.

```
A=[0 1;-1/6 5/6]; B=[0; 1]; C=[-1 5]; D=0;
[num,den]=ss2tf(A,B,C,D);
k=0:25;
u=ones(1:length(k));
y=filter(num,den,u);
stem(k,y)
```
⊙

13.6-2 The $\mathcal{Z}$-Transform Solution

The z-transform of Eq. (13.91) is given by

$$z\mathbf{X}[z] - z\mathbf{x}[0] = \mathbf{A}\mathbf{X}[z] + \mathbf{B}\mathbf{F}[z]$$

Therefore

$$(z\mathbf{I} - \mathbf{A})\mathbf{X}[z]) = z\mathbf{x}[0] + \mathbf{B}\mathbf{F}[z]$$

and

$$\begin{aligned}\mathbf{X}[z] &= (z\mathbf{I} - \mathbf{A})^{-1}z\mathbf{x}[0] + (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{F}[z] \\ &= (\mathbf{I} - z^{-1}\mathbf{A})^{-1}\mathbf{x}[0] + (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{F}[z] \qquad (13.101a)\end{aligned}$$

Hence

$$\mathbf{x}[k] = \underbrace{\mathcal{Z}^{-1}[(\mathbf{I} - z^{-1}\mathbf{A})^{-1}]\mathbf{x}[0]}_{\text{zero-input component}} + \underbrace{\mathcal{Z}^{-1}[(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{F}[z]]}_{\text{zero-state component}} \qquad (13.101b)$$

A comparison of Eq. (13.101b) with Eq. (13.93b) shows that

$$\mathbf{A}^k = \mathcal{Z}^{-1}[(\mathbf{I} - z^{-1}\mathbf{A})^{-1}] \qquad (13.102)$$

The output equation is given by

$$\begin{aligned}\mathbf{Y}[z] &= \mathbf{C}\mathbf{X}[z] + \mathbf{D}\mathbf{F}[z] \\ &= \mathbf{C}[(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\mathbf{x}[0] + (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{F}[z]] + \mathbf{D}\mathbf{F}[z] \\ &= \mathbf{C}(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\mathbf{x}[0] + [\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{F}[z] \qquad (13.103a) \\ &= \underbrace{\mathbf{C}(\mathbf{I} - z^{-1}\mathbf{A})^{-1}\mathbf{x}[0]}_{\text{zero-input response}} + \underbrace{\mathbf{H}[z]\mathbf{F}[z]}_{\text{zero-state response}}\end{aligned}$$

where

$$\mathbf{H}[z] = \mathbf{C}(z\mathbf{I} - \mathbf{A}^{-1})\mathbf{B} + \mathbf{D} \qquad (13.103b)$$

Note that $\mathbf{H}[z]$ is the transfer function matrix of the system, and $H_{ij}[z]$, the ijth element of $\mathbf{H}[z]$, is the transfer function relating the output $y_i(k)$ to the input $f_j(k)$. If we define $\mathbf{h}[k]$ as

$$\mathbf{h}[k] = \mathcal{Z}^{-1}[\mathbf{H}[z]]$$

then $\mathbf{h}[k]$ represents the unit impulse function response matrix of the system. Thus, $h_{ij}[k]$, the ijth element of $\mathbf{h}(k)$, represents the zero-state response $y_i(k)$ when the input $f_j(k) = \delta[k]$ and all other inputs are zero.

■ **Example 13.13**

Using the z-transform, find the response $y[k]$ for the system in Example 13.12. According to Eq. (13.103a)

$$\mathbf{Y}[z] = [-1 \quad 5]\begin{bmatrix} 1 & -\frac{1}{z} \\ \frac{1}{6z} & 1-\frac{5}{6z}\end{bmatrix}^{-1}\begin{bmatrix}2\\3\end{bmatrix} + [-1 \quad 5]\begin{bmatrix} z & -1 \\ \frac{1}{6} & z-\frac{5}{6}\end{bmatrix}^{-1}\begin{bmatrix}0\\ \frac{z}{z-1}\end{bmatrix}$$

$$= [-1 \quad 5]\begin{bmatrix} \frac{z(6z-5)}{6z^2-5z+1} & \frac{6z}{6z^2-5z+1} \\ \frac{-z}{6z^2-5z+1} & \frac{6z^2}{6z^2-5z+1}\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix} + [-1 \quad 5]\begin{bmatrix} \frac{z}{(z-1)(z^2-\frac{5}{6}z+\frac{1}{6})} \\ \frac{z^2}{(z-1)(z^2-\frac{5}{6}z+\frac{1}{6})}\end{bmatrix}$$

$$= \frac{13z^2-3z}{z^2-\frac{5}{6}z+\frac{1}{6}} + \frac{(5z-1)z}{(z-1)(z^2-\frac{5}{6}z+\frac{1}{6})}$$

$$= \frac{-8z}{z-\frac{1}{3}} + \frac{21z}{z-\frac{1}{2}} + \frac{12z}{z-1} + \frac{12z}{z-1} + \frac{6z}{z-\frac{1}{3}} - \frac{18z}{z-\frac{1}{2}}$$

Therefore

$$y[k] = [\underbrace{-8(3)^{-k} + 21(2)^{-k}}_{\text{zero-input response}} + \underbrace{12 + 6(3)^{-k} - 18(2)^{-k}}_{\text{zero-state response}}]u[k]$$ ■

Linear Transformation, Controllability, and Observability

The procedure for linear transformation is parallel to that in the continuous-time case (Sec. 13.4). If $\mathbf{w}$ is the transformed-state vector given by

$$\mathbf{w} = \mathbf{P}\mathbf{x}$$

then

$$\mathbf{w}[k+1] = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{w}[k] + \mathbf{P}\mathbf{B}\mathbf{f}$$

and

$$\mathbf{y}[k] = (\mathbf{C}\mathbf{P}^{-1})\mathbf{w} + \mathbf{D}\mathbf{f}$$

Controllability and observability may be investigated by diagonalizing the matrix.

13.7 Summary

An nth-order system can be described in terms of n key variables—the state variables of the system. The state variables are not unique, but can be selected in a variety of ways. Every possible system output can be expressed as a linear combination of the state variables and the inputs. Therefore the state variables describe the entire system, not merely the relationship between certain input(s) and output(s). For this reason, the state variable description is an internal description of the system. Such a description is therefore the most general system description, and it contains the information of the external descriptions, such as the impulse response and the transfer function. State-variable description can also be extended to time-varying parameter systems and nonlinear systems. An external system description may not describe a system completely.

The state equations of a system can be written directly from the knowledge of the system structure, from the system equations, or from the block diagram representation of the system. State equations consist of a set of n first-order differential equations and can be solved by time-domain or frequency-domain (transform) methods. Because a set of state variables is not unique, we can have a variety of state-space descriptions of the same system. It is possible to transform one given set of state variables into another by a linear transformation. Using such a transformation, we can see clearly which of the system states are controllable and which are observable.

References

1. Kailath, Thomas, *Linear Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1980.
2. Zadeh, L., and C. Desoer, *Linear System Theory*, McGraw-Hill, New York, 1963.

Problems

13.1-1 Convert each of the following second-order differential equations into a set of two first-order differential equations (state equations). State which of the sets represent nonlinear equations.

(a) $\ddot{y} + 10\dot{y} + 2y = f$
(b) $\ddot{y} + 2e^y\dot{y} + \log y = f$
(c) $\ddot{y} + \phi_1(y)\dot{y} + \phi_2(y)y = f$

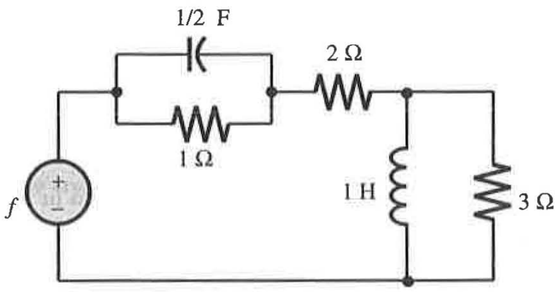


Fig. P13.2-1

13.2-1 Write the state equations for the *RLC* network in Fig. P13.2-1.

13.2-2 Write the state and output equations for the network in Fig. P13.2-2.

13.2-3 Write the state and output equations for the network in Fig. P13.2-3.

13.2-4 Write the state and output equations for the electrical network in Fig. P13.2-4.

13.2-5 Write the state and output equations for the network in Fig. P13.2-5.