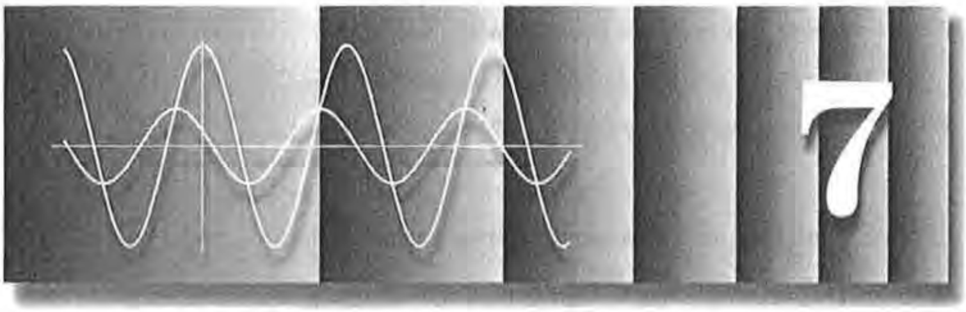


Frequency Response and Analog Filters

Filtering is an important area of signal processing. We have already discussed ideal filters in Chapter 4. In this chapter, we shall discuss the practical filter characteristics and their design. Filtering characteristics of a filter are indicated by its response to sinusoids of various frequencies varying from 0 to ∞. Such characteristics are called the frequency response of the filter. Let us start with determining the frequency response of an LTIC system.

Recall that for $h(t)$, we use the notation $H(\omega)$ for its Fourier transform and $H(s)$ for its Laplace transform. Also, when the system is causal and asymptotically stable, all the poles of $H(s)$ lie in the LHP. Hence, the region of convergence for $H(s)$ includes the $j\omega$ axis, and we can obtain the Fourier transform $H(\omega)$ by substituting $s = j\omega$ in the corresponding Laplace transform $H(s)$ (see P. 370). Therefore, $H(j\omega)$ and $H(\omega)$ represent the same entity when the system is asymptotically stable. In this and later chapters, we shall often find it convenient to use the notation $H(j\omega)$ instead of $H(\omega)$.

7.1 Frequency Response of an LTIC System

In this section we find the system response to sinusoidal inputs. In Sec. 2.4-3 we showed that an LTIC system response to an everlasting exponential input $f(t) = e^{st}$ is also an everlasting exponential $H(s)e^{st}$. As before, we represent an input-output pair using an arrow directed from the input to the output as

$$e^{st} \Longrightarrow H(s)e^{st} \tag{7.1}$$

Setting $s = \pm j\omega$ in this relationship yields

$$e^{j\omega t} \Longrightarrow H(j\omega)e^{j\omega t} \tag{7.2a}$$

$$e^{-j\omega t} \Longrightarrow H(-j\omega)e^{-j\omega t} \tag{7.2b}$$

Addition of these two relationships yields

$$2\cos\omega t \Longrightarrow H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t} = 2\text{Re}\ \left[H(j\omega)e^{j\omega t}\right] \tag{7.3}$$

We can express $H(j\omega)$ in the polar form as

$$H(j\omega) = |H(j\omega)|\, e^{j\angle H(j\omega)} \tag{7.4}$$

With this result, the relationship (7.3) becomes

$$\cos \omega t \Longrightarrow |H(j\omega)| \cos [\omega t + \angle H(j\omega)]$$

In other words, the system response $y(t)$ to a sinusoidal input $\cos \omega t$ is given by

$$y(t) = |H(j\omega)| \cos [\omega t + \angle H(j\omega)] \tag{7.5a}$$

Using a similar argument, we can show that the system response to a sinusoid $\cos(\omega t + \theta)$ is

$$y(t) = |H(j\omega)| \cos [\omega t + \theta + \angle H(j\omega)] \tag{7.5b}$$

This result, where we have let $s = j\omega$, is valid only for asymptotically stable systems because the relationship (7.1) applies only for the values of s lying in the region of convergence for $H(s)$. For the case of unstable and marginally stable systems, this region does not include the imaginary axis $s = j\omega$ (see also the footnote on p. 243).

Equation (7.5) shows that for a sinusoidal input of radian frequency ω, the system response is also a sinusoid of the same frequency ω. *The amplitude of the output sinusoid is $|H(j\omega)|$ times the input amplitude, and the phase of the output sinusoid is shifted by $\angle H(j\omega)$ with respect to the input phase* (see Fig. 7.1). For instance, if a certain system has $|H(j10)| = 3$ and $\angle H(j10) = -30°$, then the system amplifies a sinusoid of frequency $\omega = 10$ by a factor of 3 and delays its phase by 30°. The system response to an input $5\cos(10t + 50°)$ is $3 \times 5\cos(10t + 50° - 30°) = 15\cos(10t + 20°)$.

Clearly $|H(j\omega)|$ is the system **gain**, and a plot of $|H(j\omega)|$ versus ω shows the system gain as a function of frequency ω. This function is more commonly known as the **amplitude response**. Similarly, $\angle H(j\omega)$ is the **phase response** and a plot of $\angle H(j\omega)$ versus ω shows how the system modifies or changes the phase of the input sinusoid. These two plots together, as functions of ω, are called the **frequency response of the system**. Observe that $H(j\omega)$ has the information of $|H(j\omega)|$ and $\angle H(j\omega)$. For this reason, $H(j\omega)$ is also called the **frequency response of the system**. The frequency response plots show at a glance how a system responds to sinusoids of various frequencies. Thus, the frequency response of a system represents its filtering characteristic.

■ **Example 7.1**

Find the frequency response (amplitude and phase response) of a system whose transfer function is

$$H(s) = \frac{s + 0.1}{s + 5}$$

Also, find the system response $y(t)$ if the input $f(t)$ is **(a)** $\cos 2t$ **(b)** $\cos(10t - 50°)$.

In this case

$$H(j\omega) = \frac{j\omega + 0.1}{j\omega + 5}$$

Recall that the magnitude of a complex number is equal to the square root of the sum of the squares of its real and imaginary parts. Hence

$$|H(j\omega)| = \frac{\sqrt{\omega^2 + 0.01}}{\sqrt{\omega^2 + 25}} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$$

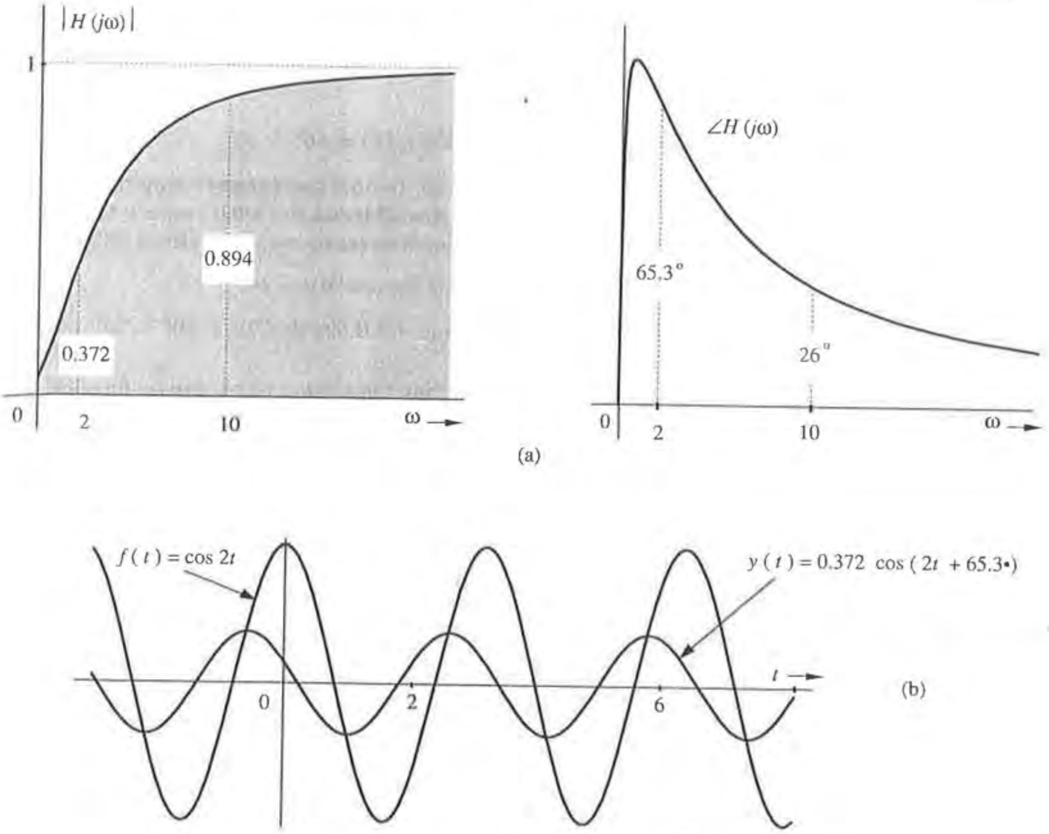


Fig. 7.1 Frequency response of an LTIC system in Example 7.1.

Both the amplitude and the phase response are depicted in Fig. 7.1a as functions of ω. These plots furnish the complete information about the frequency response of the system to sinusoidal inputs.

(a) For the input $f(t) = \cos 2t$, $\omega = 2$, and

$$|H(j2)| = \frac{\sqrt{(2)^2 + 0.01}}{\sqrt{(2)^2 + 25}} = 0.372$$

$$\angle H(j2) = \tan^{-1}\left(\frac{2}{0.1}\right) - \tan^{-1}\left(\frac{2}{5}\right) = 87.1^\circ - 21.8^\circ = 65.3^\circ$$

We also could have read these values directly from the frequency response plots in Fig. 7.1a corresponding to $\omega = 2$. This result means that for a sinusoidal input with frequency $\omega = 2$, the amplitude gain of the system is 0.372, and the phase shift is 65.3°. In other words, the output amplitude is 0.372 times the input amplitude, and the phase of the output is shifted with respect to that of the input by 65.3°. Therefore, the system response to an input cos $2t$ is

$$y(t) = 0.372 \cos(2t + 65.3^\circ)$$

The input cos $2t$ and the corresponding system response $0.372\cos(2t + 65.34^\circ)$ are illustrated in Fig. 7.1b.

(b) For the input $\cos(10t - 50°)$, instead of computing the values $|H(j\omega)|$ and $\angle H(j\omega)$ as in part (a), we shall read them directly from the frequency response plots in Fig. 7.1a corresponding to $\omega = 10$. These are:

$$|H(j10)| = 0.894 \qquad \text{and} \qquad \angle H(j10) = 26°$$

Therefore, for a sinusoidal input of frequency $\omega = 10$, the output sinusoid amplitude is 0.894 times the input amplitude and the output sinusoid is shifted with respect to the input sinusoid by 26°. Therefore, $y(t)$, the system response to an input $\cos(10t - 50°)$, is

$$y(t) = 0.894 \cos(10t - 50° + 26°) = 0.894 \cos(10t - 24°)$$

If the input were $\sin(10t - 50°)$, the response would be $0.894 \sin(10t - 50° + 26°) = 0.894 \sin(10t - 24°)$.

The frequency response plots in Fig. 7.1a show that the system has highpass filtering characteristics; it responds well to sinusoids of higher frequencies (ω well above 5), and suppresses sinusoids of lower frequencies (ω well below 5). ■

⊙ **Computer Example C7.1**

Plot the frequency response of the transfer functions $H(s) = \frac{s+5}{s^2+3s+2}$.

```
num=[1 5];
den=[1 3 2];
w=.1:.01:100;
axis([log10(.1) log10(100) -50 50])
[mag,phase,w]=bode(num,den,w);
subplot(211),semilogx(w,20*log10(mag))
subplot(212),semilogx(w,phase)
```
⊙

■ **Example 7.2**

Find and sketch the frequency response (amplitude and phase response) for

(a) an ideal delay of T seconds;

(b) an ideal differentiator;

(c) an ideal integrator.

(a) **Ideal delay of T seconds**: The transfer function of an ideal delay is [see Eq. (6.54)]

$$H(s) = e^{-sT}$$

Therefore

$$H(j\omega) = e^{-j\omega T}$$

Consequently

$$|H(j\omega)| = 1 \qquad \text{and} \qquad \angle H(j\omega) = -\omega T \tag{7.6}$$

This amplitude and phase response is shown in Fig. 7.2a. The amplitude response is constant (unity) for all frequencies. The phase shift increases linearly with frequency with a slope of $-T$. This result can be explained physically by recognizing that if a sinusoid $\cos \omega t$ is passed through an ideal delay of T seconds, the output is $\cos \omega(t - T)$. The output sinusoid amplitude is the same as that of the input for all values of ω. Therefore, the amplitude response (gain) is unity for all frequencies. Moreover, the output $\cos \omega(t - T) = \cos(\omega t - \omega T)$ has a phase shift $-\omega T$ with respect to the input $\cos \omega t$. Therefore, the phase response is linearly proportional to the frequency ω with a slope $-T$.

(b) **An ideal differentiator**: The transfer function of an ideal differentiator is [see Eq. (6.55)]

$$H(s) = s$$

Therefore

$$H(j\omega) = j\omega = \omega e^{j\pi/2}$$

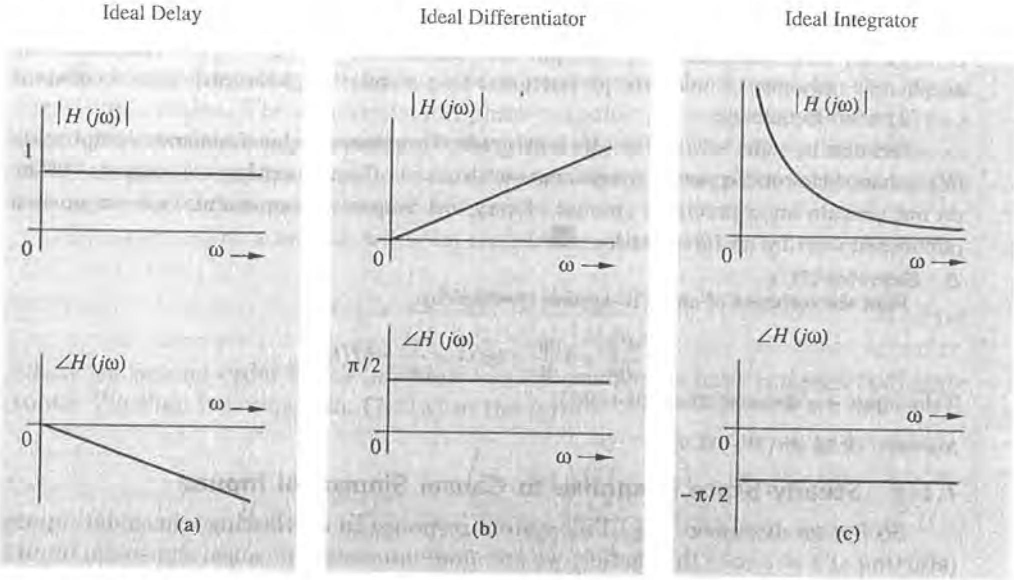


Fig. 7.2 Frequency response of an ideal (a) delay (b) differentiator (c) integrator.

Consequently

$$|H(j\omega)| = \omega \qquad \text{and} \qquad \angle H(j\omega) = \tfrac{\pi}{2} \tag{7.7}$$

This amplitude and phase response is depicted in Fig. 7.2b. The amplitude response increases linearly with frequency, and phase response is constant ($\pi/2$) for all frequencies. This result can be explained physically by recognizing that if a sinusoid $\cos \omega t$ is passed through an ideal differentiator, the output is $-\omega \sin \omega t = \omega \cos(\omega t + \frac{\pi}{2})$. Therefore, the output sinusoid amplitude is ω times the input amplitude; that is, the amplitude response (gain) increases linearly with frequency ω. Moreover, the output sinusoid undergoes a phase shift $\frac{\pi}{2}$ with respect to the input $\cos \omega t$. Therefore, the phase response is constant ($\pi/2$) with frequency.

In an ideal differentiator, the amplitude response (gain) is proportional to frequency $[|H(j\omega)| = \omega]$, so that the higher-frequency components are enhanced (see Fig. 7.2b). All practical signals are contaminated with noise, which, by its nature, is a broad-band (rapidly varying) signal containing components of very high frequencies. A differentiator can increase the noise disproportionately to the point of drowning out the desired signal. This is the reason why ideal differentiators are avoided in practice.

(c) An ideal integrator: The transfer function of an ideal integrator is [see Eq. (6.56)]

$$H(s) = \frac{1}{s}$$

Therefore

$$H(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega} = \frac{1}{\omega} e^{-j\pi/2}$$

Consequently

$$|H(j\omega)| = \frac{1}{\omega} \qquad \text{and} \qquad \angle H(j\omega) = -\frac{\pi}{2} \tag{7.8}$$

This amplitude and phase response is illustrated in Fig. 7.2c. The amplitude response is inversely proportional to frequency, and the phase shift is constant ($-\pi/2$) with frequency.

This result can be explained physically by recognizing that if a sinusoid $\cos \omega t$ is passed through an ideal integrator, the output is $\frac{1}{\omega} \sin \omega t = \frac{1}{\omega} \cos(\omega t - \frac{\pi}{2})$. Therefore, the amplitude response is inversely proportional to ω, and the phase response is constant $(-\pi/2)$ with frequency.

Because its gain is $1/\omega$, the ideal integrator suppresses higher-frequency components but enhances lower-frequency components with $\omega < 1$. Consequently, noise signals (if they do not contain an appreciable amount of very low frequency components) are suppressed (smoothed out) by an integrator. ■

△ **Exercise E7.1**

Find the response of an LTIC system specified by

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y(t) = \frac{df}{dt} + 5f(t)$$

if the input is a sinusoid $20 \sin(3t + 35°)$

Answer: $10.23 \sin(3t - 61.91°)$ ▽

7.1-1 Steady-State Response to Causal Sinusoidal Inputs

So far we discussed the LTIC system response to everlasting sinusoidal inputs (starting at $t = -\infty$). In practice, we are more interested in causal sinusoidal inputs (sinusoids starting at $t = 0$). Consider the input $e^{j\omega t} u(t)$, which starts at $t = 0$ rather than at $t = -\infty$. In this case $F(s) = 1/(s + j\omega)$. Moreover, according to Eq. (6.51) $H(s) = P(s)/Q(s)$, where $Q(s)$ is the characteristic polynomial given by $Q(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)$. Hence,

$$Y(s) = F(s)H(s) = \frac{P(s)}{(s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n)(s - j\omega)}$$

In the partial fraction expansion of the right-hand side, let the coefficients corresponding to the n terms $(s - \lambda_1)$, $(s - \lambda_2) \cdots (s - \lambda_n)$ be $k_1, k_2, \cdots k_n$. The coefficient corresponding to the last term $(s - j\omega)$ is $P(s)/Q(s)|_{s=j\omega} = H(j\omega)$. Hence,

$$Y(s) = \sum_{i=1}^{n} \frac{k_i}{s - \lambda_i} + \frac{H(j\omega)}{s - j\omega}$$

and

$$y(t) = \underbrace{\sum_{i=1}^{n} k_i e^{\lambda_i t} u(t)}_{\text{transient component } y_{tr}(t)} + \underbrace{H(j\omega)e^{j\omega t} u(t)}_{\text{steady-state component } y_{ss}(t)} \tag{7.9}$$

For an asymptotically stable system, the characteristic mode terms $e^{\lambda_i t}$ decay with time, and, therefore, constitute the so-called **transient** component of the response. The last term $H(j\omega)e^{j\omega t}$ persists forever, and is the **steady-state** component of the response given by

$$y_{ss}(t) = H(j\omega)e^{j\omega t} u(t)$$

From the argument that led to Eq. (7.5a), it follows that for a causal sinusoidal input $\cos \omega t$, the steady-state response $y_{ss}(t)$ is given by

$$y_{ss}(t) = |H(j\omega)| \cos[\omega t + \angle H(j\omega)]u(t) \tag{7.10}$$

In summary, $|H(j\omega)| \cos[\omega t + \angle H(j\omega)]$ is the total response to everlasting sinusoid $\cos \omega t$, and is also the steady-state response to the same input applied at $t = 0$.

7.2 Bode Plots

Sketching frequency response plots is considerably facilitated by the use of logarithmic scales. The amplitude and phase response plots as a function of ω on a logarithmic scale are known as the **Bode plots**. By using the asymptotic behavior of the amplitude and the phase response, we can sketch these plots with remarkable ease, even for higher-order transfer functions.

Let us consider a system with the transfer function

$$H(s) = \frac{K(s+a_1)(s+a_2)}{s(s+b_1)(s^2+b_2 s+b_3)} \tag{7.11a}$$

where the second-order factor $(s^2+b_2s+b_3)$ is assumed to have complex conjugate roots. We shall rearrange Eq. (7.11a) in the form

$$H(s) = \frac{K a_1 a_2}{b_1 b_3} \frac{\left(\frac{s}{a_1}+1\right)\left(\frac{s}{a_2}+1\right)}{s\left(\frac{s}{b_1}+1\right)\left(\frac{s^2}{b_3}+\frac{b_2}{b_3}s+1\right)} \tag{7.11b}$$

and

$$H(j\omega) = \frac{K a_1 a_2}{b_1 b_3} \frac{\left(1+\frac{j\omega}{a_1}\right)\left(1+\frac{j\omega}{a_2}\right)}{j\omega\left(1+\frac{j\omega}{b_1}\right)\left[1+j\frac{b_2\omega}{b_3}+\frac{(j\omega)^2}{b_3}\right]} \tag{7.11c}$$

This equation shows that $H(j\omega)$ is a complex function of ω. The amplitude response $|H(j\omega)|$ and the phase response $\angle H(j\omega)$ are given by

$$|H(j\omega)| = \frac{K a_1 a_2}{b_1 b_3} \frac{\left|1+\frac{j\omega}{a_1}\right|\left|1+\frac{j\omega}{a_2}\right|}{|j\omega|\left|1+\frac{j\omega}{b_1}\right|\left|1+j\frac{b_2\omega}{b_3}+\frac{(j\omega)^2}{b_3}\right|} \tag{7.12a}$$

and

$$\begin{aligned}\angle H(j\omega) = {} & \angle\left(1+\frac{j\omega}{a_1}\right)+\angle\left(1+\frac{j\omega}{a_2}\right) - \angle j\omega \\ & - \angle\left(1+\frac{j\omega}{b_1}\right) - \angle\left[1+\frac{jb_2\omega}{b_3}+\frac{(j\omega)^2}{b_3}\right]\end{aligned} \tag{7.12b}$$

From Eq. (7.12b) we see that the phase function consists of the addition of only three kinds of terms: (i) the phase of $j\omega$, which is 90° for all values of ω, (ii) the phase for the first-order term of the form $1+\frac{j\omega}{a}$, and (iii) the phase of the second-order term

$$\left[1+\frac{jb_2\omega}{b_3}+\frac{(j\omega)^2}{b_3}\right]$$

We can plot these three basic phase functions for ω in the range 0 to ∞ and then, using these plots, we can construct the phase function of any transfer by properly

Consequently

$$Me^{j\alpha(\omega)} = \frac{x + jy}{1 + x + jy}$$

Straightforward manipulation of this equation yields

$$\left(x + \frac{M^2}{M^2 - 1}\right)^2 + y^2 = \frac{M^2}{(M^2 - 1)^2}$$

This is an equation of a circle centered at $[-\frac{M^2}{M^2-1} \quad 0]$ and of radius $\frac{M}{M^2-1}$ in the $KG(j\omega)$ plane. Figure 7.12a shows circles for various values of M. Because M is the closed-loop system amplitude response, these circles are the contours of constant amplitude response of the closed-loop system. For example, the point $A = -2 - j1.85$ lies on the circle $M = 1.3$. This means, at a frequency where the open-loop transfer function is $G(j\omega) = -2 - j1.85$, the corresponding closed-loop transfer function amplitude response is 1.3.†

To obtain the closed-loop frequency response, we superimpose on these contours the Nyquist plot of the open-loop transfer function $KG(j\omega)$. For each point of $KG(j\omega)$, we can determine the corresponding value of M, the closed-loop amplitude response. From similar contours for constant α (the closed-loop phase response), we can determine the closed loop phase response. Thus, the complete closed-loop frequency response can be obtained from this plot. We are primarily interested in finding M_p, the peak value of M and ω_p, the frequency where it occurs. Figure 7.12b indicates how these values may be determined. The circle to which the Nyquist plot is tangent corresponds to M_p, and the frequency at which the Nyquist plot is tangential to this circle is ω_p. For the system, whose Nyquist plot appears in Fig. 7.12b, $M_p = 1.6$ and $\omega_p = 2$. From these values, we can estimate ζ and ω_n, and determine the transient parameters PO, t_r and t_s.

In designing systems, we first determine M_p and ω_p required to meet the given transient specifications from Eqs. (7.27). The Nyquist plot in conjunction with M circles suggests how these values of M_p and ω_p may be realized. In many cases, a mere change in gain K of the open-loop transfer function will suffice. Increasing K expands the Nyquist plot and changes the values M_p and ω_p correspondingly. If this is not enough, we should consider some form of compensation such as lag and/or lead networks. Using a computer, one can quickly observe the effect of particular form of compensation on M_p and ω_p.

7.4 Filter Design by Placement of Poles and Zeros OF H(s)

In this section we explore the strong dependence of frequency response on the location of poles and zeros of $H(s)$. This dependence points to a simple intuitive procedure to filter design. A systematic filter design procedure to meet given specifications is discussed later in Secs. 7.5, 7.6, and 7.7.

7.4-1 Dependence of Frequency Response on poles and Zeros of $H(s)$

Frequency response of a system is basically the information about the filtering capability of the system. We now examine the close connection that exists between

†We can find similar contours for constant α (the closed-loop phase response).

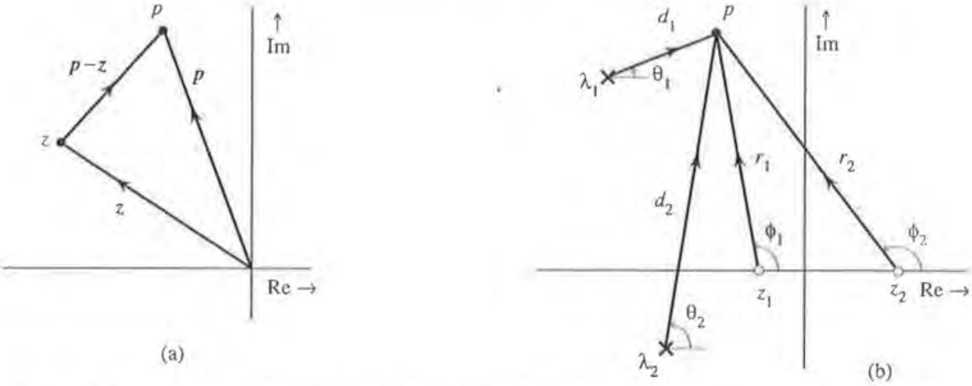


Fig. 7.13 (a)vector representation of complex numbers (b) vector representation of factors of $H(s)$.

the pole-zero locations of a system transfer function and its frequency response (or filtering characteristics). A system transfer function can be expressed as

$$H(s) = \frac{P(s)}{Q(s)} = b_n \frac{(s - z_1)(s - z_2)\cdots(s - z_n)}{(s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n)} \tag{7.28a}$$

where $z_1, z_2, \ldots, z_n$ are the zeros of $H(s)$ and the characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the poles of $H(s)$. Now the value of the transfer function $H(s)$ at some frequency $s = p$ is

$$H(s)|_{s=p} = b_n \frac{(p - z_1)(p - z_2)\cdots(p - z_n)}{(p - \lambda_1)(p - \lambda_2)\cdots(p - \lambda_n)} \tag{7.28b}$$

This equation consists of factors of the form $p - z_i$ and $p - \lambda_i$. The factor $p - z$ is a complex number represented by a vector drawn from point z to the point p in the complex plane, as illustrated in Fig. 7.13a. The length of this line segment is $|p - z|$, the magnitude of $p - z$. The angle of this directed line segment (with horizontal axis) is $\angle(p - z)$. To compute $H(s)$ at $s = p$, we draw line segments from all poles and zeros of $H(s)$ to the point p, as shown in Fig. 7.13b. The vector connecting a zero z_i to the point p is $p - z_i$. Let the length of this vector be r_i, and let its angle with the horizontal axis be ϕ_i. Then $p - z_i = r_i e^{j\phi_i}$. Similarly, the vector connecting a pole λ_i to the point p is $p - \lambda_i = d_i e^{j\theta_i}$, where d_i and θ_i are the length and the angle (with the horizontal axis), respectively, of the vector $p - \lambda_i$. Now from Eq. (7.28b) it follows that

$$\begin{aligned} H(s)|_{s=p} &= b_n \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2})\cdots(r_n e^{j\phi_n})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2})\cdots(d_n e^{j\theta_1})} \\ &= b_n \frac{r_1 r_2 \cdots r_n}{d_1 d_2 \cdots d_n} e^{j[(\phi_1 + \phi_2 + \cdots + \phi_n) - (\theta_1 + \theta_2 + \cdots + \theta_n)]} \end{aligned}$$

Therefore

$$\begin{aligned} |H(s)|_{s=p} &= b_n \frac{r_1 r_2 \cdots r_n}{d_1 d_2 \cdots d_n} \\ &= b_n \frac{\text{product of the distances of zeros to } p}{\text{product of the distances of poles to } p} \end{aligned} \tag{7.29a}$$

and

$$\angle H(s)|_{s=p} = (\phi_1 + \phi_2 + \cdots + \phi_n) - (\theta_1 + \theta_2 + \cdots + \theta_n)$$
$$= \text{sum of zero angles to } p - \text{sum of pole angles to } p \quad (7.29b)$$

Using this procedure, we can determine $H(s)$ for any value of s. To compute the frequency response $H(j\omega)$, we use $s = j\omega$ (a point on the imaginary axis), connect all poles and zeros to the point $j\omega$, and determine $|H(j\omega)|$ and $\angle H(j\omega)$ from Eqs. (7.29). We repeat this procedure for all values of ω from 0 to ∞ to obtain the frequency response.

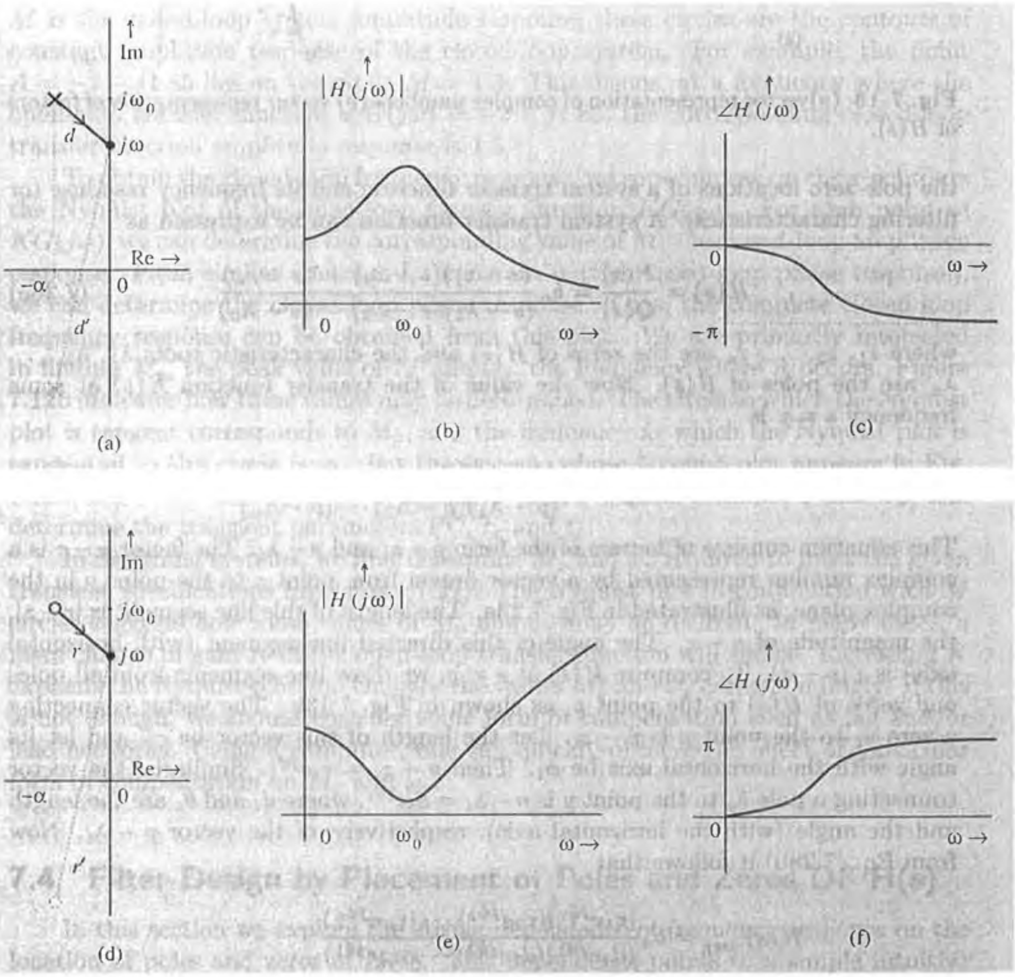


Fig. 7.14 The role of poles and zeros in determining the frequency response of an LTIC system.

Gain Enhancement by a Pole

To understand the effect of poles and zeros on the frequency response, consider a hypothetical case of a single pole $-\alpha + j\omega_0$, as depicted in Fig. 7.14a. To find the amplitude response $|H(j\omega)|$ for a certain value of ω, we connect the pole to the point $j\omega$ (Fig. 7.14a). If the length of this line is d, then $|H(j\omega)|$ is proportional to $1/d$.

$$|H(j\omega)| = \frac{K}{d} \tag{7.30}$$

where the exact value of constant K is not important at this point. As ω increases from zero up, d decreases progressively until ω reaches the value ω_0. As ω increases beyond ω_0, d increases progressively. Therefore, according to Eq. (7.30), the amplitude response $|H(j\omega)|$ increases from $\omega = 0$ until $\omega = \omega_0$, and it decreases continuously as ω increases beyond ω_0, as illustrated in Fig. 7.14b. Therefore, a pole at $-\alpha + j\omega_0$ results in a frequency-selective behavior that enhances the gain at the frequency ω_0 (resonance). Moreover, as the pole moves closer to the imaginary axis (as α is reduced), this enhancement (resonance) becomes more pronounced. This is because α, the distance between the pole and $j\omega_0$ (d corresponding to $j\omega_0$), becomes smaller, which increases the gain K/d. In the extreme case, when $\alpha = 0$ (pole on the imaginary axis), the gain at ω_0 goes to infinity. Repeated poles further enhance the frequency-selective effect. To summarize, we can enhance a gain at a frequency ω_0 by placing a pole opposite the point $j\omega_0$. The closer the pole is to $j\omega_0$, the higher is the gain at ω_0, and the gain variation is more rapid (more frequency-selective) in the vicinity of frequency ω_0. Note that a pole must be placed in the LHP for stability.

Here we have considered the effect of a single complex pole on the system gain. For a real system, a complex pole $-\alpha + j\omega_0$ must accompany its conjugate $-\alpha - j\omega_0$. We can readily show that the presence of the conjugate pole does not appreciably change the frequency-selective behavior in the vicinity of ω_0. This is because the gain in this case is K/dd', where d' is the distance of a point $j\omega$ from the conjugate pole $-\alpha - j\omega_0$. Because the conjugate pole is far from $j\omega_0$, there is no dramatic change in the length d' as ω varies in the vicinity of ω_0. There is a gradual increase in the value of d' as ω increases, which leaves the frequency-selective behavior as it was originally, with only minor changes.

Gain Suppression by a Zero

Using the same argument, we observe that zeros at $-\alpha \pm j\omega_0$ (Fig. 7.14d) will have exactly the opposite effect of suppressing the gain in the vicinity of ω_0, as shown in Fig. 7.14e). A zero on the imaginary axis at $j\omega_0$ will totally suppress the gain (zero gain) at frequency ω_0. Repeated zeros will further enhance the effect. Also, a closely-placed pair of a pole and a zero (dipole) tend to cancel out each other's influence on the frequency response. Clearly, a proper placement of poles and zeros can yield a variety of frequency-selective behavior. Using these observations, we can design lowpass, highpass, bandpass, and bandstop (or notch) filters.

Phase response can also be computed graphically. In Fig. 7.14a, angles formed by the complex conjugate poles $-\alpha \pm j\omega_0$ at $\omega = 0$ (the origin) are equal and opposite. As ω increases from 0 up, the angle θ_1 because of the pole $-\alpha + j\omega_0$, which has a negative value at $\omega = 0$, is reduced in magnitude; the angle θ_2 because of the pole $-\alpha - j\omega_0$, which has a positive value at $\omega = 0$, increases in magnitude. As a result, $\theta_1 + \theta_2$, the sum of the two angles, increases continuously, approaching a value π as $\omega \to \infty$. The resulting phase response $\angle H(j\omega) = -(\theta_1 + \theta_2)$ is illustrated in Fig. 7.14c. Similar arguments apply to zeros at $-\alpha \pm j\omega_0$. The resulting phase response $\angle H(j\omega) = (\phi_1 + \phi_2)$ is depicted in Fig. 7.14f.

We now focus on simple filters, using the intuitive insights gained in this discussion. The discussion is essentially qualitative.

7.4-2 Lowpass Filters

A typical lowpass filter has a maximum gain at $\omega = 0$. Therefore, we need to place a pole (or poles) on the real axis opposite the origin ($j\omega = 0$), as shown in Fig. 7.15a. The transfer function of this system is

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

We have chosen the numerator of $H(s)$ to be ω_c in order to normalize the dc gain $H(0)$ to unity. If d is the distance from the pole $-\omega_c$ to a point $j\omega$ (Fig. 7.15a), then

$$|H(j\omega)| = \frac{\omega_c}{d}$$

with $H(0) = 1$. As ω increases, d increases and $|H(j\omega)|$ decreases monotonically with ω, as illustrated in Fig. 7.15d by label $n = 1$. This is clearly a lowpass filter with gain enhanced in the vicinity of $\omega = 0$.

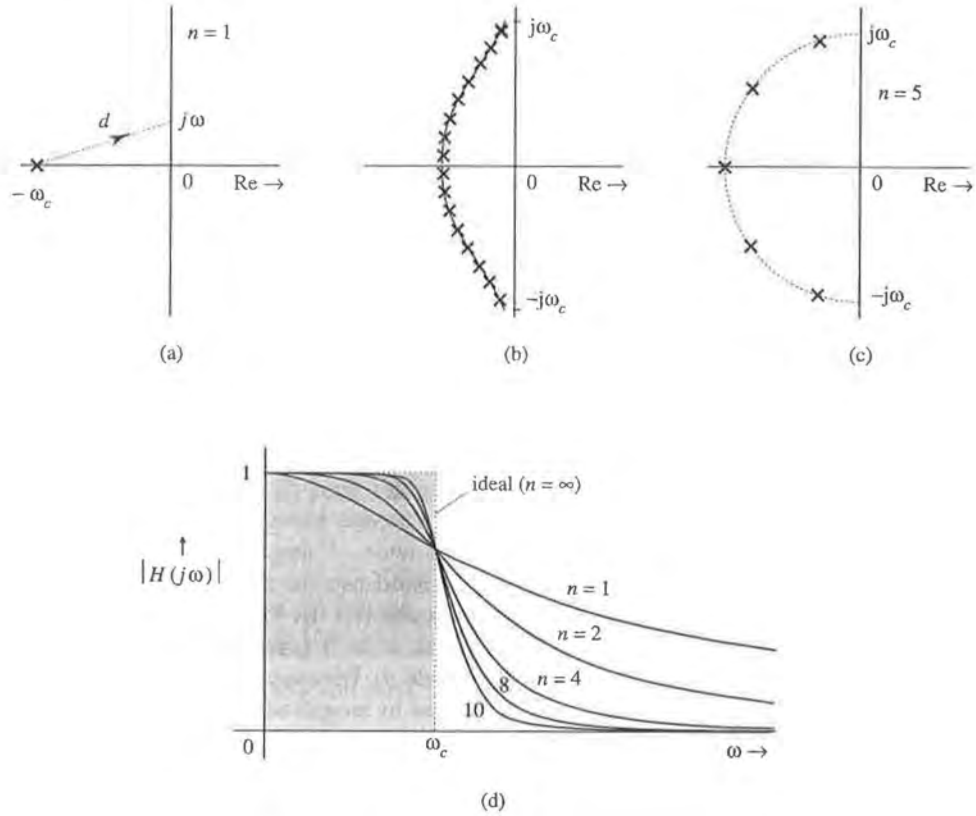


Fig. 7.15 Pole-zero configuration and the amplitude response of a lowpass (Butterworth) filter.

Wall-to-wall Poles

An ideal lowpass filter characteristic (shaded) in Fig. 7.15d, has a constant gain of unity up to frequency ω_c. Then the gain drops suddenly to 0 for $\omega > \omega_c$. To achieve the ideal lowpass characteristic, we need enhanced gain over the entire frequency band from 0 to ω_c. We know that to enhance a gain at any frequency ω, we need to place a pole opposite ω. To achieve an enhanced gain for all frequencies over the band (0 to ω_c), we need to place a pole opposite every frequency in this band. In other words, we need a **continuous wall of poles** facing the imaginary axis opposite the frequency band 0 to ω_c (and from 0 to $-\omega_c$ for conjugate poles), as depicted in Fig. 7.15b. At this point, the optimum shape of this wall is not obvious because our arguments are qualitative and intuitive. Yet, it is certain that to have enhanced gain (constant gain) at every frequency over this range, we need an infinite number of poles on this wall. We can show that for a maximally flat† response over the frequency range (0 to ω_c), the wall is a semicircle with an infinite number of poles uniformly distributed along the wall.[1] In practice, we compromise by using a finite number (n) of poles with less-than-ideal characteristics. Figure 7.15c shows the pole configuration for a fifth-order ($n = 5$) filter. The amplitude response for various values of n are illustrated in Fig. 7.15d. As $n \to \infty$, the filter response approaches the ideal. This family of filters is known as the **Butterworth** filters. There are also other families. In **Chebyshev** filters, the wall shape is a semiellipse rather than a semicircle. The characteristics of a Chebyshev filter are inferior to those of Butterworth over the passband (0, ω_c), where the characteristics show a rippling effect instead of the maximally flat response of Butterworth. But in the stopband ($\omega > \omega_c$), Chebyshev behavior is superior in the sense that Chebyshev filter gain drops faster than that of the Butterworth.

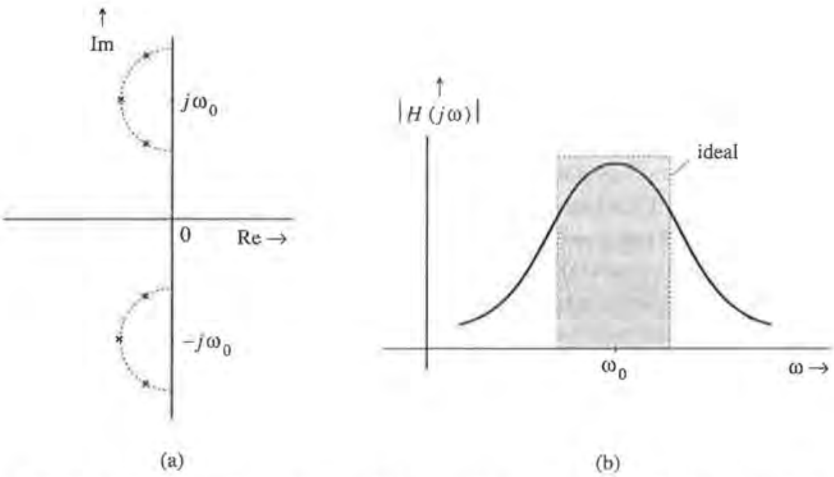


Fig. 7.16 Pole-zero configuration and the amplitude response of a bandpass filter.

7.4-3 Bandpass Filters

The shaded characteristic in Fig. 7.16b shows the ideal bandpass filter gain. In the bandpass filter, the gain is enhanced over the entire passband. Our earlier

†Maximally flat amplitude response means the first $2n-1$ derivatives of $|H(j\omega)|$ are zero at $\omega = 0$.

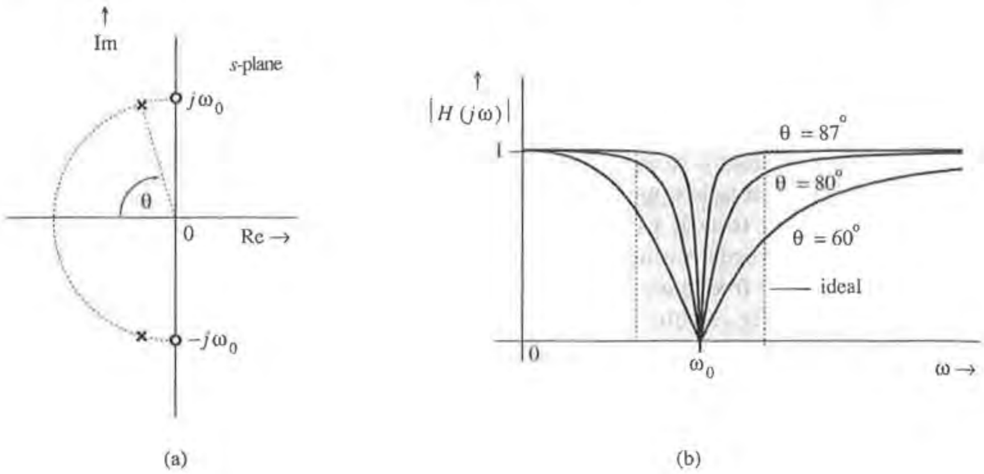


Fig. 7.17 Pole-zero configuration and the amplitude response of a bandstop (notch) filter.

discussion indicates that this can be realized by a wall of poles opposite the imaginary axis in front of the passband centered at ω_0. (There is also a wall of conjugate poles opposite $-\omega_0$.) Ideally, an infinite number of poles is required. In practice, we compromise by using a finite number of poles and accepting less-than-ideal characteristics (Fig. 7.16).

7.4-4 Notch (Bandstop) Filters

An ideal notch filter amplitude response (shown shaded in Fig. 7.17b) is a complement of the amplitude response of an ideal bandpass filter. Its gain is zero over a small band centered at some frequency ω_0 and is unity over the remaining frequencies. Realization of such a characteristic requires an infinite number of poles and zeros. Let us consider a practical second-order notch filter to obtain zero gain at a frequency $\omega = \omega_0$. For this purpose we must have zeros at $\pm j\omega_0$. The requirement of unity gain at $\omega = \infty$ requires the number of poles to be equal to the number of zeros ($m = n$). This ensures that for very large values of ω, the product of the distances of poles from ω will be equal to the product of the distances of zeros from ω. Moreover, unity gain at $\omega = 0$ requires a pole and the corresponding zero to be equidistant from the origin. For example, if we use two (complex conjugate) zeros, we must have two poles; the distance from the origin of the poles and of the zeros should be the same . This requirement can be met by placing the two conjugate poles on the semicircle of radius ω_0, as depicted in Fig. 7.17a. The poles can be anywhere on the semicircle to satisfy the equidistance condition. Let the two conjugate poles be at angles $\pm\theta$ with respect to the negative real axis. Recall that a pole and a zero in the vicinity tend to cancel out each other's influences. Therefore, placing poles closer to zeros (selecting θ closer to $\pi/2$) results in a rapid recovery of the gain from value 0 to 1 as we move away from ω_0 in either direction. Figure 7.17b shows the gain $|H(j\omega)|$ for three different values of θ.

■ **Example 7.5**

Design a second-order notch filter to suppress 60 Hz hum in a radio receiver.

We use the poles and zeros in Fig. 7.17a with $\omega_0 = 120\pi$. The zeros are at $s = \pm j\omega_0$. The two poles are at $-\omega_0 \cos\theta \pm j\omega_0 \sin\theta$. The filter transfer function is (with $\omega_0 = 120\pi$)

$$H(s) = \frac{(s - j\omega_0)(s + j\omega_0)}{(s + \omega_0 \cos\theta + j\omega_0 \sin\theta)(s + \omega_0 \cos\theta - j\omega_0 \sin\theta)}$$

$$= \frac{s^2 + \omega_0^2}{s^2 + (2\omega_0 \cos\theta)s + \omega_0^2} = \frac{s^2 + 142122.3}{s^2 + (753.98 \cos\theta)s + 142122.3}$$

and

$$|H(j\omega)| = \frac{-\omega^2 + 142122.3}{\sqrt{(-\omega^2 + 142122.3)^2 + (753.98\omega \cos\theta)^2}}$$

The closer the poles are to the zeros (closer the θ to $\frac{\pi}{2}$), the faster the gain recovery from 0 to 1 on either side of $\omega_0 = 120\pi$. Figure 7.17b shows the amplitude response for three different values of θ. This example is a case of very simple design. To achieve zero gain over a band, we need an infinite number of poles as well as of zeros. ■

Computer Example C7.3

Plot the amplitude response of the transfer function

$$H(s) = \frac{s^2 + \omega_0^2}{s^2 + (2\omega_0 \cos\theta)s + \omega_0^2}$$

of a second order notch filter for $\omega_0 = 120\pi$ and $\theta = 60^\circ, 80^\circ$, and 87°.

```
w0=120*pi;
theta=[60 80 87]*(pi/180);
for m=1:length(theta)
    num=[1 0 w0^2];
    den=[1 2*w0*cos(theta(m)) w0^2];
    w=0:.5:1000; w=w';
    [mag,phase,w]=bode(num,den,w);
    plot(w,mag),hold on,axis([0 1000 0 1.1])
end
```

Figures 7.16b and 7.17b show that a notch (stopband) filter frequency response is a complement of the bandpass filter frequency response. If $H_{BP}(s)$ and $H_{BS}(s)$ are the transfer functions of a bandpass and a bandstop filter (both centered at the same frequency), then

$$H_{BS}(s) = 1 - H_{BP}(s)$$

Therefore, a bandstop filter transfer function may also be obtained from the corresponding bandpass filter transfer function. The case of lowpass and highpass filters is similar. If $H_{LP}(s)$ and $H_{HP}(s)$ are the transfer functions of a lowpass and a highpass filter respectively (both with the same cutoff frequency), then

$$H_{HP}(s) = 1 - H_{LP}(s)$$

Therefore, a highpass filter transfer function may also be obtained from the corresponding lowpass filter transfer function.

△ **Exercise E7.2**

Using the qualitative method of sketching the frequency response, show that the system with the pole-zero configuration in Fig. 7.18a is a highpass filter, and that with the configuration in Fig. 7.18b is a bandpass filter. ▽

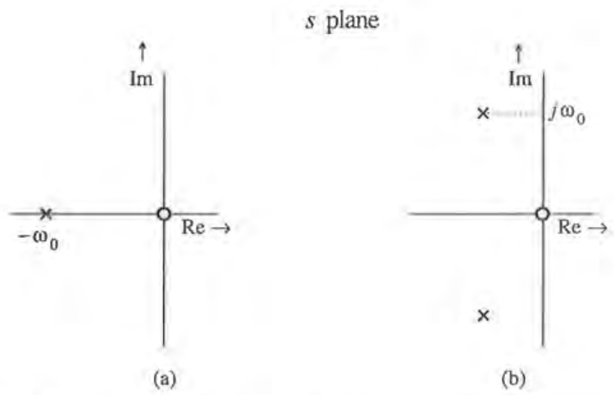


Fig. 7.18 Pole-zero configuration of a highpass filter in Exercise E7.2.

7.4-5 Practical Filters and Their Specifications

For ideal filters everything is black and white; the gains are either zero or unity over certain bands. As we saw earlier, real life does not permit such a world view. Things have to be gray or shades of gray. In practice, we can realize a variety of filter characteristics which can only approach ideal characteristics.

An ideal filter has a passband (unity gain) and a stopband (zero gain) with a sudden transition from the passband to the stopband. There is no transition band. For practical (or realizable) filters, on the other hand, the transition from the passband to the stopband (or vice versa) is gradual, and takes place over a finite band of frequencies. Moreover, for realizable filters, the gain cannot be zero over a finite band (Paley-Wiener condition). As a result, there can no true stopband for practical filters. We therefore define a **stopband** to be a band over which the gain is below some small number G_s, as illustrated in Fig. 7.19. Similarly, we define a **passband** to be a band over which the gain is between 1 and some number G_p ($G_p < 1$), as shown in Fig. 7.19. We have selected the passband gain of unity for convenience. It could be any constant. Usually the gains are specified in terms of decibels. This is simply 20 times the log (to base 10) of the gain. Thus

$$\hat{G}(\text{dB}) = 20 \log_{10} G$$

A gain of unity is 0 dB and a gain of $\sqrt{2}$ is 3.01 dB, usually approximated by 3 dB. Sometimes the specification may be in terms of attenuation, which is the negative of the gain in dB. Thus a gain of $1/\sqrt{2}$; that is, 0.707, is -3 dB, but is an attenuation of 3 dB.

In our design procedure we assume that G_p (**minimum passband gain**) and G_s (**maximum stopband gain**) are specified. Figure 7.19 shows the passband, the stopband, and the transition band for typical lowpass, bandpass, highpass, and

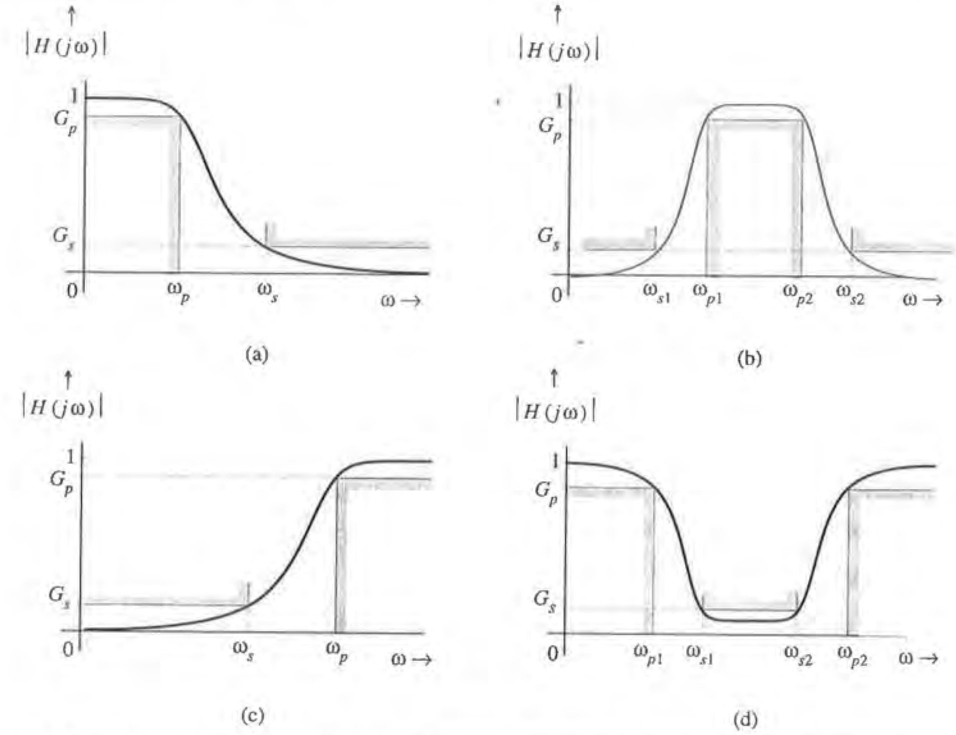


Fig. 7.19 Passband, stopband, and transitionband in various types of filters.

bandstop filters. In this chapter we shall discuss the design procedures for these four types of filters. Fortunately, the highpass, bandpass, and bandstop filters can be obtained from a basic lowpass filter by simple frequency transformations. For example, replacing s with ω_c/s in the lowpass filter transfer function results in a highpass filter. Similarly, other frequency transformations yield the bandpass and bandstop filters. Hence, it is necessary to develop a design procedure only for a basic lowpass filter. Using appropriate transformations, we can then design other types of filters. We shall consider here two well known families of filters: the Butterworth and the Chebyshev filters.

7.5 Butterworth Filters

The amplitude response $|H(j\omega)|$ of an nth order Butterworth lowpass filter is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}} \tag{7.31}$$

Observe that at $\omega = 0$, the gain $|H(j0)|$ is unity and at $\omega = \omega_c$, the gain $|H(j\omega_c)| = 1/\sqrt{2}$ or -3 dB. The gain drops by a factor $\sqrt{2}$ at $\omega = \omega_c$. Because the power is proportional to the amplitude squared, the power ratio (output power to input

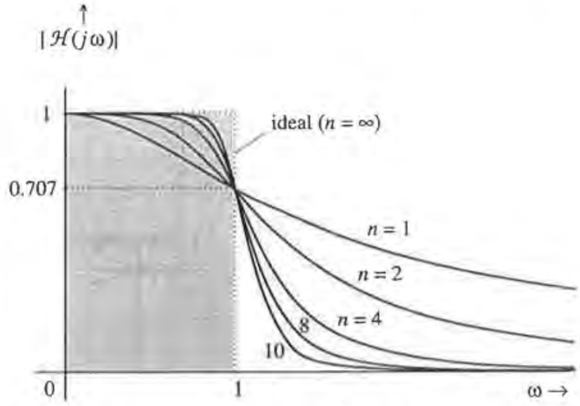


Fig. 7.20 Amplitude response of a normalized lowpass Butterworth filter.

power) drops by a factor 2 at $\omega = \omega_c$. For this reason ω_c is called the **half-power frequency** or the **3 dB-cutoff frequency** (amplitude ratio of $\sqrt{2}$ is 3 dB).

Normalized Filter

In the design procedure it proves most convenient to consider a normalized filter $\mathcal{H}(s)$, whose half-power frequency is 1 rad/s ($\omega_c = 1$). For such a filter, the amplitude characteristic in Eq. (7.31) reduces to

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}} \tag{7.32}$$

We can prepare a table of normalized transfer functions $\mathcal{H}(s)$ which yield the frequency response in Eq. (7.32) for various values of n. Once the normalized transfer function is obtained, we can obtain the desired transfer function $H(s)$ for any value of ω_c by simple frequency scaling, where we replace s by s/ω_c in the normalized $\mathcal{H}(s)$.

The amplitude response $|\mathcal{H}(j\omega)|$ of the normalized lowpass Butterworth filters is depicted in Fig. 7.20 for various values of n. From Fig. 7.20 we observe the following:

1. The Butterworth amplitude response decreases monotonically. Moreover, the first $2n - 1$ derivatives of the amplitude response are zero at $\omega = 0$. For this reason this characteristic is called **maximally flat** at $\omega = 0$. Observe that a constant characteristic (ideal) is maximally flat for all $\omega < 1$. In the Butterworth filter we try to retain this property at least at the origin.†
2. The filter gain is 1 (0 dB) at $\omega = 0$ and 0.707 (-3 dB) at $\omega = 1$ for all n. Therefore, the 3-dB (or half power) bandwidth is 1 rad/s for all n.
3. For large n, the amplitude response approaches the ideal characteristic.

To determine the corresponding transfer function $\mathcal{H}(s)$, recall that $\mathcal{H}(-j\omega)$ is the complex conjugate of $\mathcal{H}(j\omega)$. Therefore

$$\mathcal{H}(j\omega)\mathcal{H}(-j\omega) = |\mathcal{H}(j\omega)|^2 = \frac{1}{1+\omega^{2n}}$$

†Butterworth filter exhibits maximally flat characteristic also at $\omega = \infty$.

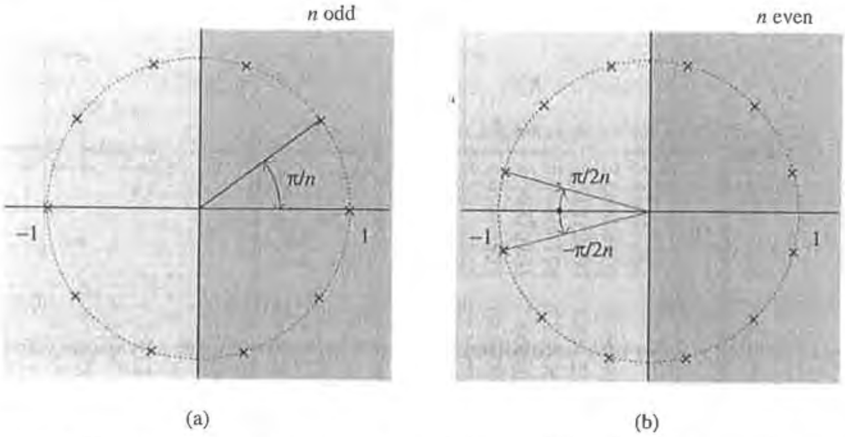


Fig. 7.21 Poles of a normalized even-order lowpass Butterworth Filter transfer function and its conjugate.

Substituting $s = j\omega$ in this equation, we obtain

$$\mathcal{H}(s)\mathcal{H}(-s) = \frac{1}{1 + (s/j)^{2n}}$$

The poles of $\mathcal{H}(s)\mathcal{H}(-s)$ are given by

$$s^{2n} = -(j)^{2n}$$

In this result we use the fact that $-1 = e^{j\pi(2k-1)}$ for integral values of k, and $j = e^{j\pi/2}$ to obtain

$$s^{2n} = e^{j\pi(2k-1+n)} \qquad k \text{ integer}$$

This equation yields the poles of $\mathcal{H}(s)\mathcal{H}(-s)$ as

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)} \qquad k = 1, 2, 3, \ldots, 2n \tag{7.33}$$

Observe that all poles have a unit magnitude; that is, they are located on a unit circle in the s-plane separated by angle π/n, as illustrated in Fig. 7.21 for odd and even n. Since $\mathcal{H}(s)$ is stable and causal, its poles must lie in the LHP. The poles of $\mathcal{H}(-s)$ are the mirror images of the poles of $\mathcal{H}(s)$ about the vertical axis. Hence, the poles of $\mathcal{H}(s)$ are those in the LHP and the poles of $\mathcal{H}(-s)$ are those in the RHP in Fig. 7.21. The poles corresponding to $\mathcal{H}(s)$ are obtained by setting $k = 1, 2, 3, \ldots, n$ in Eq. (7.33); that is

$$\begin{aligned} s_k &= e^{\frac{j\pi}{2n}(2k+n-1)} \\ &= \cos\frac{\pi}{2n}(2k+n-1) + j\sin\frac{\pi}{2n}(2k+n-1) \qquad k = 1, 2, 3, \ldots, n \end{aligned} \tag{7.34}$$

and $\mathcal{H}(s)$ is given by

$$\mathcal{H}(s) = \frac{1}{(s-s_1)(s-s_2)\cdots(s-s_n)} \tag{7.35}$$

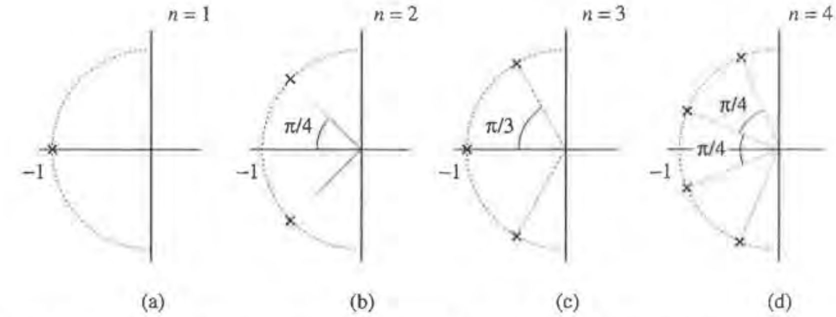


Fig. 7.22 Poles of a normalized lowpass Butterworth filter of various orders.

For instance, from Eq. (7.34), we find the poles of $\mathcal{H}(s)$ for $n = 4$ to be at angles $5\pi/8$, $7\pi/8$, $9\pi/8$, and $11\pi/8$. These lie on the unit circle, as shown in Fig. 7.22, and are given by $-0.3827 \pm j0.9239$, $-0.9239 \pm j0.3827$. Hence, $\mathcal{H}(s)$ can be expressed as

$$\begin{aligned}\mathcal{H}(s) &= \frac{1}{(s+0.3827-j0.9239)(s+0.3827+j0.9239)(s+0.9239-j0.3827)(s+0.9239+j0.3827)} \\ &= \frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)} \\ &= \frac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1}\end{aligned}$$

We can proceed in this way to find $\mathcal{H}(s)$ for any value of n. In general

$$\mathcal{H}(s) = \frac{1}{B_n(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1} \tag{7.36}$$

where $B_n(s)$ is the Butterworth polynomial of the nth order. Table 7.1 shows the coefficients $a_1, a_2, \ldots, a_{n-2}, a_{n-1}$ for various values of n; Table 7.2 shows $B_n(s)$ in factored form. In these Tables, we read for $n = 4$

$$\begin{aligned}\mathcal{H}(s) &= \frac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1} \\ &= \frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)}\end{aligned}$$

a result, which confirms our earlier computations.

We can also find the normalized Butterworth filter transfer function using MATLAB function [z,p,k]=buttap(n) to find poles,zeros and the gain factor of a normalized nth-order Butterworth filter.

⊙ **Computer Example C7.4**

Using MATLAB, find poles, zeros, and the gain factor of a normalized 4th-order Butterworth filter.

```
[z,p,k]=buttap(4)
```

MATLAB returns poles, zeros, and the gain factor k, which is unity for all orders. ⊙

Table 7.1: **Coefficients of Butterworth Polynomial** $B_n(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	1.41421356								
3	2.00000000	2.00000000							
4	2.61312593	3.41421356	2.61312593						
5	3.23606798	5.23606798	5.23606798	3.23606798					
6	3.86370331	7.46410162	9.14162017	7.46410162	3.86370331				
7	4.49395921	10.09783468	14.59179389	14.59179389	10.09783468	4.49395921			
8	5.12583090	13.13707118	21.84615097	25.68835593	21.84615097	13.13707118	5.12583090		
9	5.75877048	16.58171874	31.16343748	41.98638573	41.98638573	31.16343748	16.58171874	5.75877048	
10	6.39245322	20.43172909	42.80206107	64.88239627	74.23342926	64.88239627	42.80206107	20.43172909	6.39245322

Table 7.2: **Butterworth Polynomials in Factorized Form**

n	$B_n(s)$
1	$s+1$
2	$s^2 + 1.41421356s + 1$
3	$(s+1)(s^2+s+1)$
4	$(s^2 + 0.76536686s + 1)(s^2 + 1.84775907s + 1)$
5	$(s+1)(s^2 + 0.61803399s + 1)(s^2 + 1.931803399s + 1)$
6	$(s^2 + 0.51763809s + 1)(s^2 + 1.41421356s + 1)(s^2 + 1.93185165s + 1)$
7	$(s+1)(s^2 + 0.44504187s + 1)(s^2 + 1.24697960s + 1)(s^2 + 1.80193774s + 1)$
8	$(s^2 + 0.39018064s + 1)(s^2 + 1.11114047s + 1)(s^2 + 1.66293922s + 1)(s^2 + 1.96157056s + 1)$
9	$(s+1)(s^2 + 0.34729636s + 1)(s^2 + s + 1)(s^2 + 1.53208889s + 1)(s^2 + 1.87938524s + 1)$
10	$(s^2 + 0.31286893s + 1)(s^2 + 0.90798100s + 1)(s^2 + 1.41421356s + 1)(s^2 + 1.78201305s + 1)(s^2 + 1.97537668s + 1)$

Frequency Scaling

Although Tables 7.1 and 7.2 are for normalized Butterworth filters with 3 dB bandwidth $\omega_c = 1$, the results can be extended to any value of ω_c by simply replacing s by s/ω_c. This step implies replacing ω by ω/ω_c in Eq. (7.32). For example, the second-order Butterworth filter for $\omega_c = 100$ can be obtained from Table 7.1 by replacing s by $s/100$. This step yields

$$\begin{aligned} H(s) &= \frac{1}{\left(\frac{s}{100}\right)^2 + \sqrt{2}\left(\frac{s}{100}\right) + 1} \\ &= \frac{1}{s^2 + 100\sqrt{2}s + 10^4} \end{aligned} \tag{7.37}$$

The amplitude response $|H(j\omega)|$ of the filter in Eq. (7.37) is identical to that of normalized $|\mathcal{H}(j\omega)|$ in Eq. (7.32), expanded by a factor 100 along the horizontal (ω) axis (frequency scaling).

Determination of n, the Filter Order

If $\hat{G}_x$ is the gain of a lowpass Butterworth filter in dB units at $\omega = \omega_x$, then according to Eq. (7.31)

$$\hat{G}_x = 20\log_{10}|H(j\omega_x)| = -10\log\left[1 + \left(\frac{\omega_x}{\omega_c}\right)^{2n}\right]$$

Substitution of the specifications in Fig. 7.19a (gains $\hat{G}_p$ at ω_p and $\hat{G}_s$ at ω_s) in this equation yields

$$\hat{G}_p = -10\log\left[1 + \left(\frac{\omega_p}{\omega_c}\right)^{2n}\right]$$

$$\hat{G}_s = -10\log\left[1 + \left(\frac{\omega_s}{\omega_c}\right)^{2n}\right]$$

or

$$\left(\frac{\omega_p}{\omega_c}\right)^{2n} = 10^{-\hat{G}_p/10} - 1 \tag{7.38a}$$

$$\left(\frac{\omega_s}{\omega_c}\right)^{2n} = 10^{-\hat{G}_s/10} - 1 \tag{7.38b}$$

Dividing (7.38b) by (7.38a), we obtain

$$\left(\frac{\omega_s}{\omega_p}\right)^{2n} = \left[\frac{10^{-\hat{G}_s/10} - 1}{10^{-\hat{G}_p/10} - 1}\right]$$

and

$$n = \frac{\log\left[\left(10^{-\hat{G}_s/10} - 1\right) / \left(10^{-\hat{G}_p/10} - 1\right)\right]}{2\log(\omega_s/\omega_p)} \tag{7.39}$$

Also from Eq. (7.38a)

$$\omega_c = \frac{\omega_p}{\left[10^{-\hat{G}_p/10} - 1\right]^{1/2n}} \tag{7.40}$$

Alternatively, from Eq. (7.38b)

$$\omega_c = \frac{\omega_s}{\left[10^{-\hat{G}_s/10} - 1\right]^{1/2n}} \tag{7.41}$$

Example 7.6

Design a Butterworth lowpass filter to meet the specifications (Fig. 7.23):

(i) Passband gain to lie between 1 and $G_p = 0.794$ ($\hat{G}_p = -2$ dB) for $0 \le \omega < 10$.

(ii) Stopband gain not to exceed $G_s = 0.1$ ($\hat{G}_s = -20$ dB) for $\omega \ge 20$.

Step 1: Determine n

Here $\omega_p = 10$, $\omega_s = 20$, $\hat{G}_p = -2$ dB, and $\hat{G}_s = -20$ dB. Substituting these values in Eq. (7.39) yields

$$n = 3.701$$

Since n can only be an integer, we choose

$$n = 4$$

Step 2: Determine ω_c

Substitution of $n = 4$, $\omega_p = 10$ in Eq. (7.40) yields

$$\omega_c = 10.693$$

Alternately, substitution of $n = 4$ in Eq. (7.41) yields

$$\omega_c = 11.261$$

Because we selected $n = 4$ rather than 3.701, we obtain two different values of ω_c. Choice of $\omega_c = 10.693$ will satisfy exactly the requirement $G_p = 0.794$ over the passband (0, 10), and will surpass the requirement $G_s = 0.1$ in the stopband $\omega \ge 20$. On the other hand, choice of $\omega_c = 11.261$ will exactly satisfy the requirement on G_s but will oversatisfy the requirement for G_p. Let us choose the former case ($\omega_c = 10.693$).

Step 3: Determine the normalized transfer function $\mathcal{H}(s)$

The normalized fourth-order transfer function $\mathcal{H}(s)$ is found from Table 7.1 as

$$\mathcal{H}(s) = \frac{1}{s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1}$$

Step 4: Determine the final filter transfer function $H(s)$

The desired transfer function with $\omega_c = 10.693$ is obtained by replacing s with $s/10.693$ in the normalized transfer function $\mathcal{H}(s)$ as

$$\begin{aligned}
H(s) &= \frac{1}{\left(\frac{s}{10.693}\right)^4 + 2.6131\left(\frac{s}{10.693}\right)^3 + 3.4142\left(\frac{s}{10.692}\right)^2 + 2.6131\left(\frac{s}{10.693}\right) + 1} \\
&= \frac{13073.7}{s^4 + 27.942s^3 + 390.4s^2 + 3194.88s + 13073.7} \\
&= \frac{13073.7}{(s^2 + 8.1844s + 114.34)(s^2 + 19.758s + 114.34)}
\end{aligned}$$

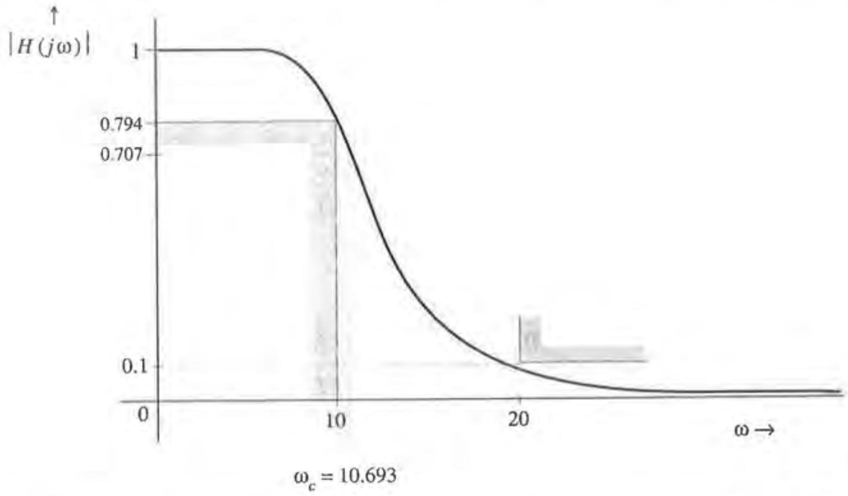


Fig. 7.23 Amplitude response of the lowpass Butterworth filter in Example 7.6.

The amplitude response of this filter is given by Eq. (7.31) with $n = 4$ and $\omega_c = 10.693$

$$|H(j\omega)| = \frac{1}{\sqrt{(\frac{\omega}{10.693})^8 + 1}}$$

Figure 7.23 shows the amplitude response of this filter.

We could also have used the alternate value of $\omega_c = 11.261$. This choice would result in a slightly different design. Either of the two designs satisfies the specifications. ■

Computer Example C7.5
Solve Example 7.6 using MATLAB

```
% Step 1: Determine n
wp=10; ws=20; Gp=-2; Gs=-20;
P1=-Gs/10; P2=-Gp/10; Wsp=ws/wp;
nc=log((10^P1-1)/(10^P2-1))/(2*log(Wsp));
n=ceil(nc);
% Step 2: Determine Wc (option that satisfies passband requirement
% exactly and may oversatisfy stopband requirement).
Wc=wp/(10^P2-1)^(1/(2*n));
% Step 3: Determine the normalized transfer function H(s)
for k=1:n
   A=(2*(k-1)+n+1)/(2*n);
   Sk=cos(A*pi)+j*sin(A*pi);
   s=[s Sk];
end
s=s';
num1=[0 1]; den1=poly([s']);
% Step 4: Determine the final filter transfer function H(s)
num2=[0 Wc^n]; den2=poly(Wc*[s']);
fprintf('Filter Order is n = %i\n',n)
fprintf('Cutoff Frequency of the Filter is Wc = %.4f\n',Wc)
disp('Poles of the transfer function are'),s
```

```
disp('The normalized fourth-order transfer function is')
printsys(abs(num1),abs(den1))
disp('The transfer function with s replaced by s/Wc is')
printsys(abs(num2),abs(den2))
% Step 5: Amplitude response of the filter
w=0:.01:40; w=w';
[mag,phase,w]=bode(num2,den2,w);
plot(w,mag)
Filter Order is n = 4
Cutoff Frequency of the Filter is Wc = 10.6934
Poles of the transfer function are
s = -0.3827 - 0.9239i
-0.9239 - 0.3827i
-0.9239 + 0.3827i
-0.3827 + 0.9239i
The normalized fourth-order transfer function is
```

$$\text{num/den} = \frac{1}{\text{s\^{}4 + 2.613 s\^{}3 + 3.414 s\^{}2 + 2.613 s + 1}}$$

The transfer function with s replaced by s/Wc is

$$\text{num/den} = \frac{13,000}{\text{s\^{}4 + 27.94 s\^{}3 + 390.4 s\^{}2 + 3195 s + 1.3 e+004}}$$

Using M-files from MATLAB Signal Processing Toolbox

We can also compute the desired filter transfer function using appropriate M-files from the *Signal Processing Toolbox* as shown in the next few examples.

Computer Example C7.6

Using M-file functions in MATLAB, design a lowpass Butterworth filter to meet the specifications in Example 7.6.

```
Wp=10;Ws=20;Gp=-2;Gs=-20;
[n,Wc]=buttord(Wp,Ws,-Gp,-Gs,'s');
[num,den]=butter(n,Wc,'s');
```

Here num and den are the coefficients of the numerator and the denominator polynomials of the desired filter. In this example, the matlab answer is $n+1$ element vectors as num= 0 0 0 0 16081 and den= 1 29 433 3732 16081; that is,

$$H(s) = \frac{16081}{s^4 + 29s^3 + 433s^2 + 3732s + 16081}$$

This is the alternate solution where the passband specifications are exceeded, but the stopband specifications are satisfied exactly. On the other hand, the solution in Example C7.5 exceeds the stopband specifications, but satisfies the passband specifications exactly because of use of Eq. (7.40) [rather than Eq. (7.41)]. To plot amplitude response, we can use the last three functions from Example C7.5.

△ **Exercise E7.3**

Determine n, the order of the lowpass Butterworth filter to meet the following specifications: $\hat{G}_p = -0.5\,\text{dB}$, $\hat{G}_s = -20\,\text{dB}$, $\omega_p = 100$, and $\omega_s = 200$.

Answer: 5. ▽

7.6 Chebyshev Filters

The amplitude response of a normalized Chebyshev lowpass filter is given by

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1+\epsilon^2 {C_n}^2(\omega)}} \tag{7.42}$$

where $C_n(\omega)$, the nth-order Chebyshev polynomial, is given by

$$C_n(\omega) = \cos\left(n \cos^{-1} \omega\right) \tag{7.43a}$$

An alternative expression for $C_n(\omega)$ is

$$C_n(\omega) = \cosh\left(n \cosh^{-1} \omega\right) \tag{7.43b}$$

The form (7.43a) is most convenient to compute $C_n(\omega)$ for $|\omega| < 1$ and form (7.43b) is convenient for computing $C_n(\omega)$ for $|\omega| > 1$. We can show[1] that $C_n(\omega)$ is also expressible in polynomial form, as shown† in Table 7.3 for $n = 1$ to 10.

The normalized Chebyshev lowpass amplitude response [Eq. (7.42)] is depicted in Fig. 7.24 for $n = 6$ and $n = 7$. We make the following general observations:

Table 7.3: Chebyshev Polynomials

n	$C_n(\omega)$
0	1
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$
7	$64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega$
8	$128\omega^8 - 256\omega^6 + 160\omega^4 - 32\omega^2 + 1$
9	$256\omega^9 - 576\omega^7 + 432\omega^5 - 120\omega^3 + 9\omega$
10	$512\omega^{10} - 1280\omega^8 + 1120\omega^6 - 400\omega^4 + 50\omega^2 - 1$

†The Chebyshev polynomial $C_n(\omega)$ has the property[1]

$$C_n(\omega) = 2\omega C_{n-1}(\omega) - C_{n-2}(\omega) \qquad n > 2$$

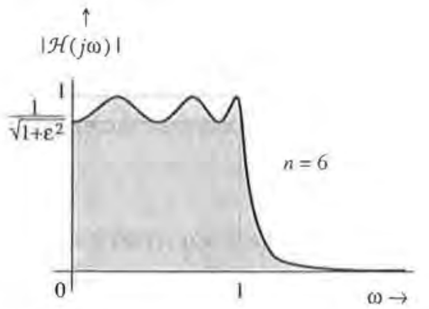
Thus, knowing that

$$C_0(\omega) = 1 \qquad \text{and} \qquad C_1(\omega) = \omega$$

we can construct $C_n(\omega)$ for any value of n. For example,

$$C_2(\omega) = 2\omega C_1(\omega) - C_0(\omega) = 2\omega^2 - 1$$

and so on.



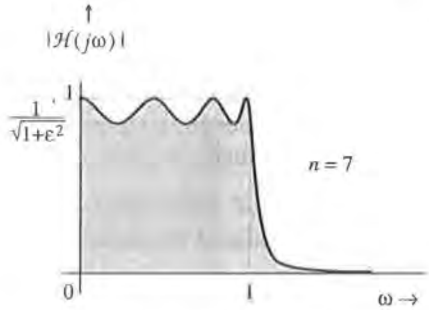


Fig. 7.24 Amplitude response of normalized sixth- and seventh-order lowpass Chebyshev filters.

1. The Chebyshev amplitude response has ripples in the passband and is smooth (monotonic) in the stopband. The passband is $0 \leq \omega \leq 1$, and there is a total of n maxima and minima over the passband $0 \leq \omega \leq 1$.

2. From Table 7.3, we observe that

$$C_n^2(0) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases} \tag{7.44}$$

Therefore, the dc gain is

$$|\mathcal{H}(0)| = \begin{cases} 1 & n \text{ odd} \\ \frac{1}{\sqrt{1+\epsilon^2}} & n \text{ even} \end{cases} \tag{7.45}$$

3. The parameter ϵ controls the height of ripples. In the passband, r, the ratio of the maximum gain to the minimum gain is

$$r = \sqrt{1+\epsilon^2} \tag{7.46a}$$

This ratio r, specified in decibels, is

$$\hat{r} = 20 \log \sqrt{1+\epsilon^2} = 10 \log_{10}(1+\epsilon^2) \tag{7.46b}$$

so that

$$\epsilon^2 = 10^{\hat{r}/10} - 1 \tag{7.47}$$

Because all the ripples in the passband are of equal height, the Chebyshev polynomials $C_n(\omega)$ are known as **equal-ripple functions**.

4. The ripple is present only over the passband $0 \leq \omega \leq 1$. At $\omega = 1$, the amplitude response is $1/\sqrt{1+\epsilon^2} = 1/r$. For $\omega > 1$, the gain decreases monotonically.

5. For Chebyshev filters, the ripple $\hat{r}$ dB takes the place of $\hat{G}_p$ (the minimum gain in the passband). For example, $\hat{r} \leq 2$ dB specifies that the gain variations of more than 2 dB cannot be tolerated in the passband. In the Butterworth filter $\hat{G}_p = -2$ dB means the same thing.

6. If we reduce the ripple, the passband behavior improves, but it does so at the cost of stopband behavior. As r is decreased (ϵ is reduced), the gain in the stopband increases, and vice-versa. Hence, there is a tradeoff between the allowable passband ripple and the desired attenuation in the stopband. Note

that the extreme case $\epsilon = 0$ yields zero ripple, but the filter now becomes an allpass filter, as seen from Eq. 7.42, by letting $\epsilon = 0$.

7. Finally, the Chebyshev filter has a sharper cutoff (smaller transition band) than the same-order Butterworth filter,† but this is achieved at the expense of inferior passband behavior (rippling).

Determination of n (Filter Order)

For a normalized Chebyshev filter, the gain $\hat{G}$ in dB [see Eq. (7.42)] is

$$\hat{G} = -10 \log \left[1 + \epsilon^2 {C_n}^2(\omega)\right]$$

The gain is $\hat{G}_s$ at ω_s. Therefore

$$\hat{G}_s = -10 \log \left[1 + \epsilon^2 {C_n}^2(\omega_s)\right] \tag{7.48}$$

or

$$\epsilon^2 {C_n}^2(\omega_s) = 10^{-\hat{G}_s/10} - 1$$

Use of Eq. (7.43b) and Eq.(7.47) in the above equation yields

$$\cosh \left[n \cosh^{-1}(\omega_s)\right] = \left[\frac{10^{-\hat{G}_s/10} - 1}{10^{\hat{r}/10} - 1}\right]^{1/2}$$

Hence

$$n = \frac{1}{\cosh^{-1}(\omega_s)} \cosh^{-1} \left[\frac{10^{-\hat{G}_s/10} - 1}{10^{\hat{r}/10} - 1}\right]^{1/2} \tag{7.49a}$$

Note that these equations are for normalized filters, where $\omega_p = 1$. For a general case, we replace ω_s with $\frac{\omega_s}{\omega_p}$ to obtain

$$n = \frac{1}{\cosh^{-1}(\omega_s/\omega_p)} \cosh^{-1} \left[\frac{10^{-\hat{G}_s/10} - 1}{10^{\hat{r}/10} - 1}\right]^{1/2} \tag{7.49b}$$

Pole Locations

We could follow the procedure of the Butterworth filter to obtain the pole locations of the Chebyshev filter. The procedure is straightforward but tedious and does not yield any special insight into our development. The Butterworth filter poles lie on a semicircle. We can show that the poles of an nth-order normalized Chebyshev filter lie on a semiellipse of the major and minor semiaxes $\cosh x$ and $\sinh x$, respectively, where[1]

$$x = \frac{1}{n} \sinh^{-1} \left(\frac{1}{\epsilon}\right) \tag{7.50}$$

The Chebyshev filter poles are

$$s_k = -\sin \left[\frac{(2k-1)\pi}{2n}\right] \sinh x + j \cos \left[\frac{(2k-1)\pi}{2n}\right] \cosh x \quad k = 1, 2, \cdots, n \tag{7.51}$$

†We can show[2] that at higher frequencies (in the stopband), the Chebyshev filter gain is smaller than the comparable Butterworth filter gain by about $6(n-1)$ dB.

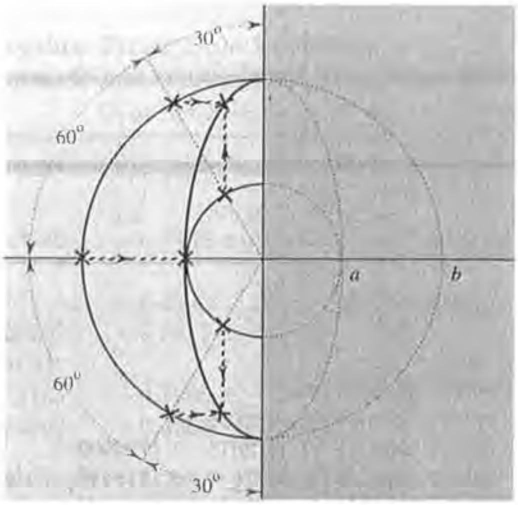


Fig. 7.25 Poles of a normalized third-order lowpass Chebyshev filter transfer function and its conjugate.

The geometrical construction for determining the pole location is depicted in Fig. 7.25 for $n = 3$. A similar procedure applies to any n; it consists of drawing two semicircles of radii $a = \sinh x$ and $b = \cosh x$. We now draw radial lines along the corresponding Butterworth angles and locate the nth-order Butterworth poles (shown by crosses) on the two circles. The location of the kth Chebyshev pole is the intersection of the horizontal projection and the vertical projection from the corresponding kth Butterworth poles on the outer and the inner circle, respectively.

The transfer function $\mathcal{H}(s)$ of the normalized nth-order lowpass Chebyshev filter is

$$\mathcal{H}(s) = \frac{K_n}{C'_n(s)} = \frac{K_n}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{7.52}$$

The constant K_n is selected to have proper dc gain, as shown in Eq. (7.45). As a result

$$K_n = \begin{cases} a_0 & n \text{ odd} \\ \dfrac{a_0}{\sqrt{1+\epsilon^2}} = \dfrac{a_0}{10^{\hat{r}/20}} & n \text{ even} \end{cases} \tag{7.53}$$

The design procedure is considerably simplified by ready-made tables of the polynomial $C'_n(s)$ in Eq. (7.52) or the pole locations of $\mathcal{H}(s)$. Table 7.4 lists the coefficients a_0, a_1, a_2, $\cdots$, a_{n-1} of the polynomial $C'_n(s)$ in Eq. (7.52) for $\hat{r} = 0.5$, 1, 2, and 3 dB ripples corresponding to the values of $\epsilon = 0.3493$, 0.5088, 0.7648, and 0.9976, respectively. Table 7.5 lists the poles of various Chebyshev filters for the same values of $\hat{r}$ (and ϵ). Tables listing more extensive values of $\hat{r}$ (or ϵ) can be found in the literature. We can also use MATLAB functions for this purpose.

⊙ **Computer Example C7.7**

Using MATLAB, find poles, zeros, and the gain factor of a normalized 3rd-order Chebyshev filter with $\hat{r} = 2$ dB.

```
[z,p,k]=cheb1ap(3,2)
```
⊙

Table 7.4: Chebyshev Filter Coefficients of the Denominator Polynomial $C'_n(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0$

n	a_0	a_1	a_2	a_3	a_4	a_5	a_6
1	2.8627752					0.5 db ripple	
2	1.5162026	1.4256245				($\hat{r} = 0.5$)	
3	0.7156938	1.5348954	1.2529130				
4	0.3790506	1.0254553	1.7168662	1.1973856			
5	0.1789234	0.7525181	1.3095747	1.9373675	1.1724909		
6	0.0947626	0.4323669	1.1718613	1.5897635	2.1718446	1.1591761	
7	0.0447309	0.2820722	0.7556511	1.6479029	1.8694079	2.4126510	1.1512176
1	1.9652267					1 db ripple	
2	1.1025103	1.0977343				($\hat{r} = 1$)	
3	0.4913067	1.2384092	0.9883412				
4	0.2756276	0.7426194	1.4539248	0.9528114			
5	0.1228267	0.5805342	0.9743961	1.6888160	0.9368201		
6	0.0689069	0.3070808	0.9393461	1.2021409	1.9308256	0.9282510	
7	0.0307066	0.2136712	0.5486192	1.3575440	1.4287930	2.1760778	0.9231228
1	1.3075603					2 db ripple	
2	0.8230604	0.8038164				($\hat{r} = 2$)	
3	0.3268901	1.0221903	0.7378216				
4	0.2057651	0.5167981	1.2564819	0.7162150			
5	0.0817225	0.4593491	0.6934770	1.4995433	0.7064606		
6	0.0514413	0.2102706	0.7714618	0.8670149	1.7458587	0.7012257	
7	0.0204228	0.1660920	0.3825056	1.1444390	1.0392203	1.9935272	0.6978929
1	1.0023773					3 db ripple	
2	0.7079478	0.6448996				($\hat{r} = 3$)	
3	0.2505943	0.9283480	0.5972404				
4	0.1769869	0.4047679	1.1691176	0.5815799			
5	0.0626391	0.4079421	0.5488626	1.4149874	0.5744296		
6	0.0442467	0.1634299	6990977	6906098	1.6628481	0.5706979	
7	0.0156621	0.1461530	0.3000167	1.0518448	0.8314411	1.9115507	0.5684201

Table 7.5: Chebyshev Filter Pole Locations

n	$\hat{r} = 0.5$	$\hat{r} = 1$	$\hat{r} = 2$	$\hat{r} = 3$
1	-2.8628	-1.9652	-1.3076	-1.0024
2	$-0.7128 \pm j1.0040$	$-0.5489 \pm j0.8951$	$-0.4019 \pm j0.8133$	$-0.3224 \pm j0.7772$
3	-0.6265	-0.4942	-0.3689	-0.2986
	$-0.3132 \pm j1.0219$	$-0.2471 \pm j0.9660$	$-0.1845 \pm j0.9231$	$-0.1493 \pm j0.9038$
4	$-0.1754 \pm j1.0163$	$-0.1395 \pm j0.9834$	$-0.1049 \pm j0.9580$	$-0.0852 \pm j0.9465$
	$-0.4233 \pm j0.4209$	$-0.3369 \pm j0.4073$	$-0.2532 \pm j0.3968$	$-0.2056 \pm j0.3920$
5	-0.3623	-0.2895	-0.2183	-0.1775
	$-0.1120 \pm j1.0116$	$-0.0895 \pm j0.9901$	$-0.0675 \pm j0.9735$	$-0.0549 \pm j0.9659$
	$-0.2931 \pm j0.6252$	$-0.2342 \pm j0.6119$	$-0.1766 \pm j0.6016$	$-0.1436 \pm j0.5970$
6	$-0.0777 \pm j1.0085$	$-0.0622 \pm j0.9934$	$-0.0470 \pm j0.9817$	$-0.0382 \pm j0.9764$
	$-0.2121 \pm j0.7382$	$-0.1699 \pm j0.7272$	$-0.1283 \pm j0.7187$	$-0.1044 \pm j0.7148$
	$-0.2898 \pm j0.2702$	$-0.2321 \pm j0.2662$	$-0.1753 \pm j0.2630$	$-0.1427 \pm j0.2616$
7	-0.2562	-0.2054	-0.1553	-0.1265
	$-0.0570 \pm j1.0064$	$-0.0457 \pm j0.9953$	$-0.0346 \pm j0.9866$	$-0.0281 \pm j0.9827$
	$-0.1597 \pm j0.8071$	$-0.1281 \pm j0.7982$	$-0.0969 \pm j0.7912$	$-0.0789 \pm j0.7881$
	$-0.2308 \pm j0.4479$	$-0.1851 \pm j0.4429$	$-0.1400 \pm j0.4391$	$-0.1140 \pm j0.4373$
8	$-0.0436 \pm j1.0050$	$-0.0350 \pm j0.9965$	$-0.0265 \pm j0.9898$	$-0.0216 \pm j0.9868$
	$-0.1242 \pm j0.8520$	$-0.0997 \pm j0.8447$	$-0.0754 \pm j0.8391$	$-0.0614 \pm j0.8365$
	$-0.1859 \pm j0.5693$	$-0.1492 \pm j0.5644$	$-0.1129 \pm j0.5607$	$-0.0920 \pm j0.5590$
	$-0.2193 \pm j0.1999$	$-0.1760 \pm j0.1982$	$-0.1332 \pm j0.1969$	$-0.1085 \pm j0.1962$
9	-0.1984	-0.1593	-0.1206	-0.0983
	$-0.0345 \pm j1.0040$	$-0.0277 \pm j0.9972$	$-0.0209 \pm j0.9919$	$-0.0171 \pm j0.9896$
	$-0.0992 \pm j0.8829$	$-0.0797 \pm j0.8769$	$-0.0603 \pm j0.8723$	$-0.0491 \pm j0.8702$
	$-0.1520 \pm j0.6553$	$-0.1221 \pm j0.6509$	$-0.0924 \pm j0.6474$	$-0.0753 \pm j0.6459$
	$-0.1864 \pm j0.3487$	$-0.1497 \pm j0.3463$	$-0.1134 \pm j0.3445$	$-0.0923 \pm j0.3437$
10	$-0.0279 \pm j1.0033$	$-0.0224 \pm j0.9978$	$-0.0170 \pm j0.9935$	$-0.0138 \pm j0.9915$
	$-0.0810 \pm j0.9051$	$-0.1013 \pm j0.7143$	$-0.0767 \pm j0.7113$	$-0.0401 \pm j0.8945$
	$-0.1261 \pm j0.7183$	$-0.0650 \pm j0.9001$	$-0.0493 \pm j0.8962$	$-0.0625 \pm j0.7099$
	$-0.1589 \pm j0.4612$	$-0.1277 \pm j0.4586$	$-0.0967 \pm j0.4567$	$-0.0788 \pm j0.4558$
	$-0.1761 \pm j0.1589$	$-0.1415 \pm j0.1580$	$-0.1072 \pm j0.1574$	$-0.0873 \pm j0.1570$

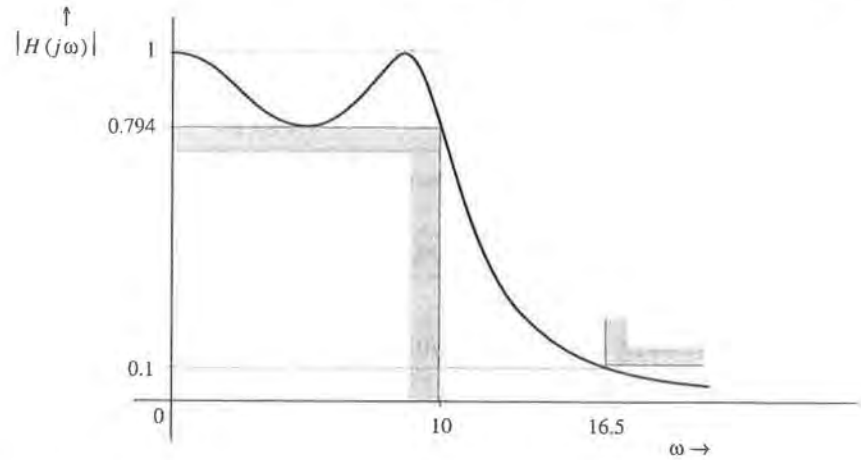


Fig. 7.26 Amplitude response of the lowpass Chebyshev filter in Example 7.7.

Example 7.7

Design a Chebyshev lowpass filter to satisfy the following criteria (Fig. 7.26):

The ratio $\hat{r} \leq 2$ dB over a passband $0 \leq \omega \leq 10$ ($\omega_p = 10$). The stopband gain $\hat{G}_s \leq -20$ dB for $\omega > 16.5$ ($\omega_s = 16.5$).

Observe that the specifications are the same as those in Example 7.6, except for the transition band. Here the transition band is from 10 to 16.5, whereas in Example 7.6 it is 10 to 20. Despite this stringent requirement, we shall find that Chebyshev requires a lower-order filter than the Butterworth filter found in Example 7.6.

Step 1: Determining n

According to Eq. (7.49b), we have

$$n = \frac{1}{\cosh^{-1}(1.65)} \cosh^{-1}\left[\frac{10^2 - 1}{10^{0.2} - 1}\right]^{1/2} = 2.999$$

Because n must be an integer, we select $n = 3$. Observe that even with more stringent requirements, the Chebyshev filter requires only $n = 3$. The passband behavior of the Butterworth filter, however, is superior (maximally flat at $\omega = 0$) compared to that of the Chebyshev, which has rippled passband characteristics.

Step 2: Determining $\mathcal{H}(s)$

We may use the Table 7.4 to determine $\mathcal{H}(s)$. For $n = 3$ and $\hat{r} = 2$ dB, we read the coefficients of the denominator polynomial of $\mathcal{H}(s)$ as $a_0 = 0.3269$, $a_1 = 1.0222$, and $a_2 = 0.7378$. Also in Eq. (7.53), for odd n, the numerator is given by a constant $K_n = a_0 = 0.3269$. Therefore,

$$\mathcal{H}(s) = \frac{0.3269}{s^3 + 0.7378s^2 + 1.0222s + 0.3269} \tag{7.54}$$

Because there are infinite possible combinations of n and $\hat{r}$, Table 7.4 (or 7.5) can list values of the denominator coefficients for values of $\hat{r}$ in quantum increments only.[1] For the values of n and $\hat{r}$ not listed in the Table, we can compute pole locations from Eq. (7.51). For the sake of demonstration, we now recompute $\mathcal{H}(s)$ using this method. In this case, the value of ϵ is [see Eq. (7.47)]

$$\epsilon = \sqrt{10^{\hat{r}/10} - 1} = \sqrt{10^{0.2} - 1} = 0.7647$$

From Eq. (7.50)

$$x = \frac{1}{n}\sinh^{-1}\frac{1}{\epsilon} = \frac{1}{3}\sinh^{-1}(1.3077) = 0.3610$$

Now from Eq. (7.51), we have $s_1 = -0.1844 + j0.9231$, $s_2 = -0.3689$, and $s_3 = -0.1844 - j0.9231$. Therefore

$$\begin{aligned}\mathcal{H}(s) &= \frac{K_n}{(s+0.3689)(s+0.1844+j0.9231)(s+0.1844-j0.9231)}\\ &= \frac{K_n}{s^3+0.7378s^2+1.0222s+0.3269} = \frac{0.3269}{s^3+0.7378s^2+1.0222s+0.3269}\end{aligned}$$

which confirms the earlier result.

Step 3: Determining $H(s)$

Recall that $\omega_p = 1$ for the normalized transfer function. For $\omega_p = 10$, the desired transfer function $H(s)$ can be obtained from the normalized transfer function $\mathcal{H}(s)$ by replacing s with $s/\omega_p = s/10$. Therefore

$$\begin{aligned}H(s) &= \frac{0.3269}{(\frac{s}{10}+0.3689)(\frac{s}{10}+0.1844+j0.9231)(\frac{s}{10}+0.1844-j0.9231)}\\ &= \frac{326.9}{s^3+7.378s^2+102.22s+326.9}\end{aligned}$$

In the present case, $\hat{r} = 2$ dB means that [see Eq. (7.47)]

$$\epsilon^2 = 10^{0.2} - 1 = 0.5849$$

The frequency response is [see Eq. (7.42) and Table 7.3]

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1+0.5849(4\omega^3-3\omega)^2}}$$

This is the normalized filter amplitude response. The actual filter response $|H(j\omega)|$ is obtained by replacing ω with $\frac{\omega}{\omega_p}$; that is, with $\frac{\omega}{10}$ in $\mathcal{H}(j\omega)$

$$\begin{aligned}|H(j\omega)| &= \frac{1}{\sqrt{1+0.5849\left[4\left(\frac{\omega}{10}\right)^3 - 3\left(\frac{\omega}{10}\right)^2\right]}}\\ &= \frac{10^3}{\sqrt{9.3584\omega^6 - 1403.76\omega^4 + 52640\omega^2 + 10^6}}\end{aligned}$$

Observe that despite more stringent specifications than those in Example 7.6, the Chebyshev filter requires $n = 3$ compared to the Butterworth filter in Example 7.6, which requires $n = 4$. Figure 7.26 shows the amplitude response. ■

⊙ Computer Example C7.8

Design a lowpass Chebyshev filter for the specifications in Example 7.7 using functions from *Signal Processing Toolbox* in MATLAB.

```
Wp=10;Ws=16.5;r=2;Gs=-20;
[n,Wp]=cheb1ord(Wp,Ws,r,-Gs,'s');
[num,den]=cheby1(n,r,Wp,'s');
```

MATLAB returns $n = 3$ and num= 0 0 0 326.8901, den= 1 7.3782 102.219 326.8901; that is,

$$H(s) = \frac{326.8901}{s^3 + 7.3782s^2 + 102.219s + 326.8901}$$

a result, which agrees with the solution in Example 7.7. To plot amplitude response, we can use the last three functions from Example C7.5. ⊙

△ **Exercise E7.4**

Determine n, the order of the lowpass Butterworth filter to meet the following specifications: $\hat{G}_p = -0.5\,\text{dB}$, $\hat{G}_s = -20\,\text{dB}$, $\omega_p = 100$, and $\omega_s = 200$.

Answer: 5. ▽

△ **Exercise E7.5**

Determine n (the order) and the transfer function of a Chebyshev filter to meet the following specifications: $\hat{r} = 2$ dB, $\hat{G}_s = -20$ dB, $\omega_p = 10$ rad/s, and $\omega_s = 28$ rad/s.

Answer: $n = 2$

$$H(s) = \frac{50.5823}{(s + 4.0191 + j6.8937)(s + 4.0191 - j6.8937)} = \frac{50.5823}{s^2 + 8.0381s + 63.6768}$$

Hint: In this case $K_2 = \frac{a_0}{\sqrt{1+\epsilon^2}}$ ▽

7.6-1 Inverse Chebyshev Filters

The passband behavior of the Chebyshev filters exhibits ripples and the stopband is smooth. Generally, passband behavior is more important and we would prefer that the passband have smooth response. However, ripples can be tolerated in the stopband as long as they meet a given specification. The **inverse Chebyshev** filter does exactly that. Both, the Butterworth and the Chebyshev filters, have finite poles and no finite zeros. The inverse Chebyshev has finite zeros and poles. It exhibits maximally flat passband response and equal-ripple stopband response.

The inverse Chebyshev response can be obtained from the Chebyshev in two steps as follows: Let $\mathcal{H}_C(\omega)$ be the Chebyshev amplitude response given in Eq. (7.42). In the first step, we subtract $|\mathcal{H}_C(\omega)|^2$ from 1 to obtain a highpass filter characteristic where the stopband (from 0 to 1) has ripples and the passband (from 1 to ∞) is smooth. In the second step, we interchange the stopband and passband by frequency transformation where ω is replaced by $1/\omega$. This step inverts the passband from the range 1 to ∞ to the range 0 to 1, and the stopband is now from 1 to ∞. Moreover, the passband is now smooth and the stopband has ripples. This is precisely the inverse Chebyshev amplitude response $|\mathcal{H}(\omega)|$ given by

$$|\mathcal{H}(\omega)|^2 = 1 - |\mathcal{H}_C(1/\omega)|^2 = \frac{\epsilon^2 C_n^2(1/\omega)}{1 + \epsilon^2 C_n^2(1/\omega)}$$

where $C_n(\omega)$ are the nth-order Chebyshev polynomials listed in Table 7.3.

The inverse Chebyshev filters are preferable to the Chebyshev filters in many ways. For example, the passband behavior, especially for small ω, is better for the inverse Chebyshev than for the Chebyshev or even for the Butterworth filter of the same order. The inverse Chebyshev also has the smallest transition band of the three filters. Moreover, the phase function (or time-delay) characteristic of the inverse Chebyshev filter is better than that of the Chebyshev filter.[2] Both the Chebyshev and inverse Chebyshev filter require the same order n to meet a given set of specifications.[1] But the inverse Chebyshev realization requires more elements and thus is less economical than the Chebyshev filter. But the inverse Chebyshev does require fewer elements than a comparable performance Butterworth filter. Rather than give complete development of inverse Chebyshev filters, we shall solve a problem using MATLAB functions from the *Signal Processing Toolbox*.

⊙ **Computer Example C7.9**

Design a lowpass inverse Chebyshev filter for the specifications in Example 7.7, using functions from *Signal Processing Toolbox* in MATLAB.

```
Wp=10;Ws=16.5;Gp=-2;Gs=-20;
[n,Ws]=cheb2ord(Wp,Ws,-Gp,-Gs,'s');
[num,den]=cheby2(n,-Gs,Ws,'s')
```

MATLAB returns $n = 3$ and num= 0 5 0 1805.9, den= 1 23.2 256.4 1805.9; that is,

$$H(s) = \frac{5s^2 + 1805.9}{s^3 + 23.2s^2 + 256.4s + 1805.9}$$

To plot amplitude response, we can use the last three functions from Example C7.5. ⊙

7.6-2 Elliptic Filters

Recall our discussion in Sec. 7.4 that placing a zero on the imaginary axis (at $s = j\omega$) causes the gain ($|H(j\omega)|$) to go to zero (infinite attenuation). We can realize a sharper cutoff characteristic by placing a zero (or zeros) near $\omega = \omega_s$. Butterworth and Chebyshev filters do not make use of zeros in $\mathcal{H}(s)$. But an elliptic filter does. This is the reason for the superiority of the elliptic filter.

A Chebyshev filter has a smaller transition band compared to that of a Butterworth filter because a Chebyshev filter allows rippling in the passband (or stopband). If we allow ripple in both the passband and the stopband, we can achieve further reduction in the transition band. Such is the case with elliptic (or Cauer) filters, whose normalized amplitude response is given by

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 {R_n}^2(\omega)}}$$

where $R_n(\omega)$ is the nth-order Chebyshev rational function determined from the specific ripple characteristic. The parameter ϵ controls the ripple. The gain at ω_p ($\omega_p = 1$ for the normalized case) is $\frac{1}{\sqrt{1+\epsilon^2}}$.

The elliptic filter is the most efficient if we can tolerate ripples in both the passband and the stopband. For a given transition band, it provides the largest ratio of the passband gain to stopband gain, or for a given ratio of passband to stopband gain, it requires the smallest transition band. In compensation, however, we must accept ripples in both the passband and the stopband. In addition, because of zeros in the numerator of $\mathcal{H}(s)$, the elliptic filter response decays at a slower rate at frequencies higher than ω_s. For instance, the amplitude response of a third-order elliptic filter decays at a rate of only -6 dB/octave at very high frequencies. This is because the filter has two zero and three poles. The two zeros increase the amplitude response at a rate of 12 dB/octave, and the three poles reduce the amplitude response at a rate of -18 dB/octave, thus giving a net decay rate of -6 dB/octave. For the Butterworth and Chebyshev filters, there are no zeros in $\mathcal{H}(s)$.

Therefore, their amplitude response decays at a rate of -18 dB/octave. However, the rate of decay of the amplitude response is seldom important as long as we meet our specification of a given G_s at ω_s.

Calculation of pole-zero locations of elliptic filters is much more complicated than that in Butterworth or even Chebyshev filters. Fortunately, this task is greatly simplified by computer programs and extensive ready-made design Tables available in the literature.[3] The MATLAB function [z,p,k]=ellipap(n,-Gp,-Gs) in *Signal Processing Toolbox* determines poles, zeros, and the gain factor of a normalized analog elliptic lowpass filter of order n with a minimum passband gain Gp dB, and maximum stopband gain Gs dB. The normalized passband edge is 1 rad/s.

Computer Example C7.10

Design the lowpass elliptic filter for the specifications in Example 7.7 using functions from *Signal Processing Toolbox* in MATLAB.

```
Wp=10;Ws=16.5;Gp=-2;Gs=-20;
[n,Wp]=ellipord(Wp,Ws,-Gp,-Gs,'s');
[num,den]=ellip(n,-Gp,-Gs,Wp,'s')
```

MATLAB returns $n = 3$ and num= 0 2.7881 0 481.1626, den= 1 7.261 106.9991 481.1626; that is,

$$H(s) = \frac{2.7881s^2 + 481.1626}{s^3 + 7.261s^2 + 106.9991s + 481.1626}$$

To plot amplitude response, we can use the last three functions from Example C7.5.

7.7 Frequency Transformations

Earlier we saw how a lowpass filter transfer function of arbitrary specifications can be obtained from a normalized lowpass filter using frequency scaling. Using certain frequency transformations, we can obtain transfer functions of highpass, bandpass, and bandstop filters from a basic lowpass filter (the prototype filter) design. For example, a highpass filter transfer function can be obtained from the prototype lowpass filter transfer function by replacing s with ω_p/s. Similar transformations allow us to design bandpass and bandstop filters from appropriate lowpass prototype filters.

The prototype filter may be of any kind, such as Butterworth, Chebyshev, elliptic, and so on. We first design a suitable prototype lowpass filter $\mathcal{H}_p(s)$. In the next step, we replace s with a proper transformation $T(s)$ to obtain the desired highpass, bandpass, or bandstop filter.

7.7-1 Highpass Filters

Figure 7.27a shows an amplitude response of a typical highpass filter. The appropriate lowpass prototype response required for the design of a highpass filter in Fig. 7.27a is depicted in Fig. 7.27b. We must first determine this prototype filter transfer function $\mathcal{H}_p(s)$ with the passband $0 \leq \omega \leq 1$ and the stopband $\omega \geq \omega_p/\omega_s$. The desired transfer function of the highpass filter to satisfy specifications in Fig. 7.27a is then obtained by replacing s with $T(s)$ in $\mathcal{H}_p(s)$, where

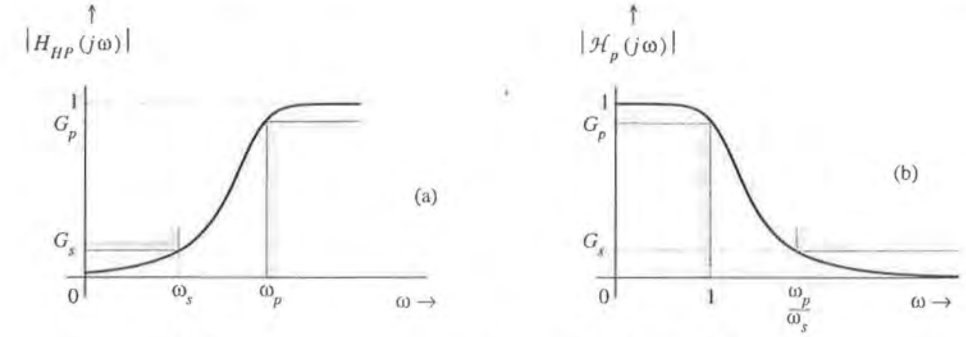


Fig. 7.27 Frequency transformation for highpass filters.

$$T(s) = \frac{\omega_p}{s} \tag{7.55}$$

■ Example 7.8

Design a Chebyshev highpass filter with the amplitude response specifications illustrated in Fig. 7.28a with $\omega_s = 100$, $\omega_p = 165$, $G_s = 0.1\,(-20\,\text{dB})$, and $G_p = 0.794\,(-2\,\text{dB})$.

Step 1: Determine the prototype lowpass filter

The prototype lowpass filter has $\hat{\omega}_p = 1$ and $\hat{\omega}_s = 165/100 = 1.65$. This means the prototype filter in Fig. 7.27b has a passband $0 \leq \omega \leq 1$ and a stopband $\omega \geq 1.65$, as shown in Fig. 7.28b. Also, $G_p = 0.794\,(-2\,\text{dB})$ and $G_s = 0.1\,(-20\,\text{dB})$. We already designed a Chebyshev filter with these specifications in Example 7.7. The transfer function of this filter is [Eq. (7.54)]

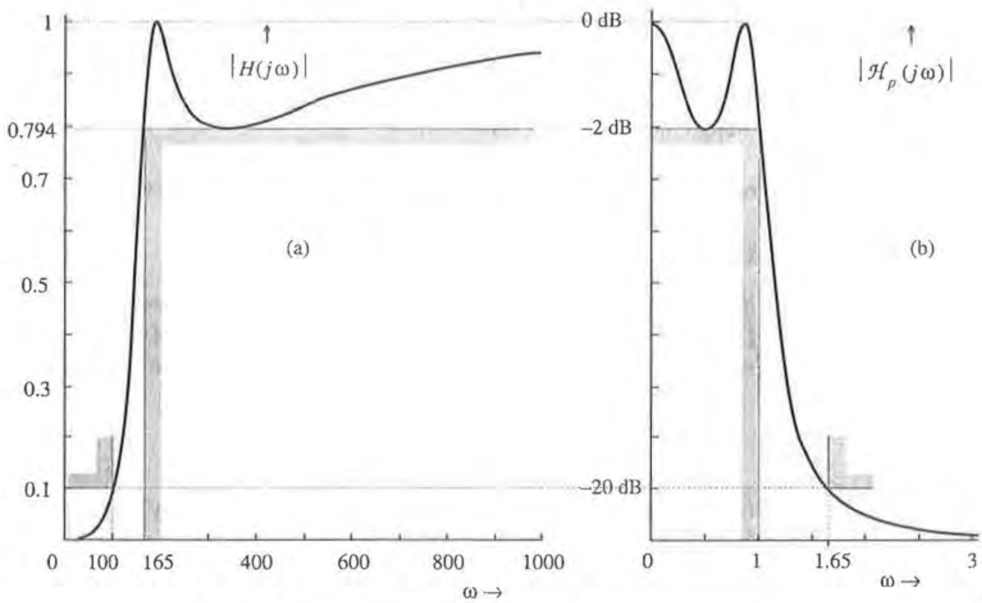


Fig. 7.28 Highpass Filter Design for Example 7.8.

$$\mathcal{H}_p(s) = \frac{0.3269}{s^3 + 0.7378s^2 + 1.0222s + 0.3269}$$

The amplitude response of this prototype filter is depicted in Fig. 7.28b.

Step 2: Substitute s with $T(s)$ in $\mathcal{H}_p(s)$

The desired highpass filter transfer function $H(s)$ is obtained from $\mathcal{H}_p(s)$ by replacing s with $T(s) = \omega_p/s = 165/s$. Therefore

$$\begin{aligned} H(s) &= \frac{0.3269}{\left(\frac{165}{s}\right)^3 + 0.7378\left(\frac{165}{s}\right)^2 + 1.0222(\frac{165}{s}) + 0.3269} \\ &= \frac{s^3}{s^3 + 515.94s^2 + 61445.75s + 13742005} \end{aligned}$$

The amplitude response $|H(j\omega)|$ for this filter is illustrated in Fig. 7.28a. ■

⊙ **Computer Example C7.11**

Design the highpass filter for the specifications in Example 7.8 using functions from *Signal Processing Toolbox* in MATLAB. We shall give here MATLAB functions for all types of filters.

```
Ws=100;Wp=165;Gp=-2;Gs=-20;
% Butterworth
[n,Wn]=buttord(Wp,Ws,-Gp,-Gs,'s')
[num,den]=butter(n,Wn,'high','s')
% Chebyshev
[n,Wn]=cheb1ord(Wp,Ws,-Gp,-Gs,'s')
[num,den]=cheby1(n,-Gp,Wn,'high','s')
% Inverse Chebyshev
[n,Wn]=cheb2ord(Wp,Ws,-Gp,-Gs,'s')
[num,den]=cheby2(n,-Gs,Wn,'high','s')
% Elliptic
[n,Wn]=ellipord(Wp,Ws,-Gp,-Gs,'s')
[num,den]=ellip(n,-Gp,-Gs,Wn,'high','s')
```

To plot amplitude response, we can use the last three functions in Example C7.5. ⊙

7.7-2 Bandpass Filters

Figure 7.29a shows an amplitude response of a typical bandpass filter. To design such a filter, we first find $\mathcal{H}_p(s)$, the transfer function of a prototype lowpass filter, to meet the specifications in Fig. 7.29b, where ω_s is given by the smaller of

$$\frac{\omega_{p_1}\omega_{p_2} - {\omega_{s_1}}^2}{\omega_{s_1}(\omega_{p_2} - \omega_{p_1})} \qquad \text{or} \qquad \frac{{\omega_{s_2}}^2 - \omega_{p_1}\omega_{p_2}}{\omega_{s_2}(\omega_{p_2} - \omega_{p_1})} \tag{7.56}$$

Now, the desired transfer function of the bandpass filter to satisfy the specifications in Fig. 7.29a is obtained from $\mathcal{H}_p(s)$ by replacing s with $T(s)$, where

$$T(s) = \frac{s^2 + \omega_{p_1}\omega_{p_2}}{(\omega_{p_2} - \omega_{p_1})s} \tag{7.57}$$

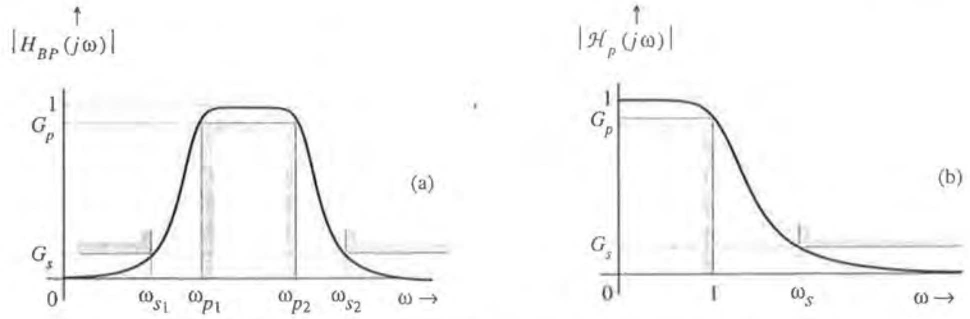


Fig. 7.29 Frequency transformation for bandpass filters.

■ Example 7.9

Design a Chebyshev bandpass filter with the amplitude response specifications shown in Fig. 7.30a with $\omega_{p_1} = 1000$, $\omega_{p_2} = 2000$, $\omega_{s_1} = 450$, $\omega_{s_2} = 4000$, $G_s = 0.1\,(-20\,\text{dB})$, and $G_p = 0.891\,(-1\,\text{dB})$. Observe that for Chebyshev filter, $G_p = -1$ dB is equivalent to $\hat{r} = 1$ dB.

The solution is executed in two steps: in the first step, we determine the lowpass prototype filter transfer function $\mathcal{H}_p(s)$. In the second step, the desired bandpass filter transfer function is obtained from $\mathcal{H}_p(s)$ by substituting s with $T(s)$, the lowpass to bandpass transformation in Eq. (7.57).

Step 1: Find $\mathcal{H}_p(s)$, the lowpass prototype filter transfer function.

This is done in 3 substeps as follows:

Step 1.1: Find ω_s for the prototype filter.

The frequency ω_s is found [using Eq. (7.56)], to be the smaller of

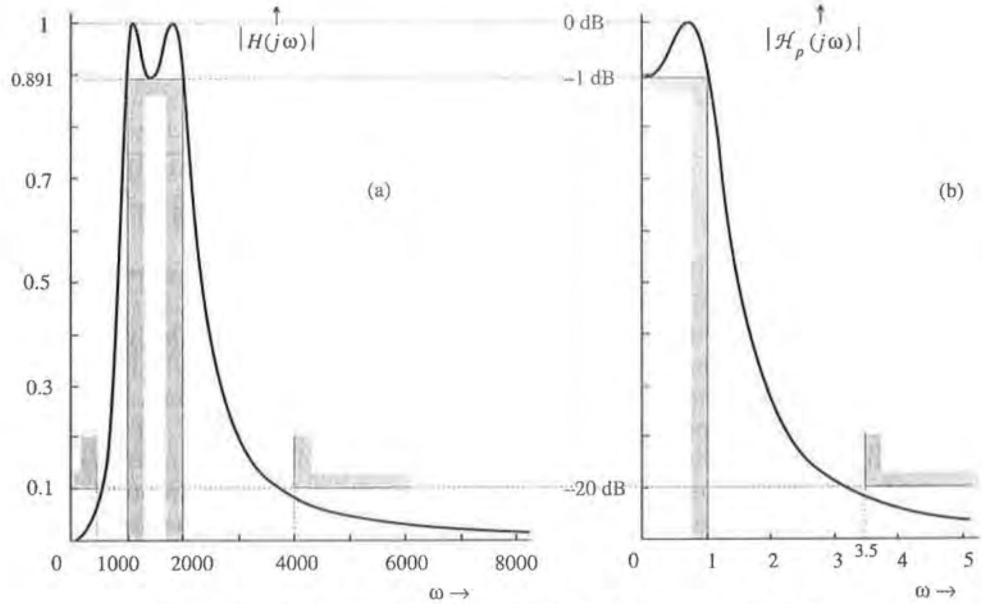


Fig. 7.30 Chebyshev Bandpass Filter Design for Example 7.9.

$$\frac{(1000)(2000)-(450)^2}{450(2000-1000)}=3.99 \quad \text{and} \quad \frac{(4000)^2-(1000)(2000)}{4000(2000-1000)}=3.5$$

which is 3.5.

Step 1.2: Determine n

We now need to design a prototype lowpass filter in Fig. 7.29b with $\hat{G}_p = -1$ dB, $\hat{G}_s = -20$ dB, $\omega_p = 1$, and $\omega_s = 3.5$, as illustrated in Fig. 7.30b. The Chebyshev filter order n required to meet these specifications is obtained from Eq. (7.49b) (or Eq. (7.49a) because, in this case, $\omega_p = 1$), as

$$n = \frac{1}{\cosh^{-1}(3.5)} \cosh^{-1}\left[\frac{10^2-1}{10^{0.1}-1}\right]^{\frac{1}{2}} = 1.904$$

a result, which is rounded up to $n = 2$.

Step 1.3: Determine the prototype filter transfer function $\mathcal{H}_p(s)$

We can obtain the transfer function of the second-order Chebyshev filter by computing its poles for $n = 2$ and $\hat{r} = 1$ ($\epsilon = 0.5088$) using Eq. (7.51). However, since Table 7.4 lists the denominator polynomial for $\hat{r} = 1$ and $n = 2$, we need not perform the computations and may use the ready-made transfer function directly as

$$\mathcal{H}_p(s) = \frac{0.9826}{s^2 + 1.0977s + 1.1025} \tag{7.58}$$

Here we used Eq. (7.53) to find the numerator $K_n = \frac{a_0}{\sqrt{1+\epsilon^2}} = \frac{1.1025}{\sqrt{1.2589}} = 0.9826$. The amplitude response of this prototype filter is depicted in Fig. 7.30b.

Step 2: Find the desired bandpass filter transfer function $H(s)$ using the lowpass to bandpass transformation.

Finally, the desired bandpass filter transfer function $H(s)$ is obtained from $\mathcal{H}_p(s)$ by replacing s with $T(s)$, where [see Eq. (7.57)]

$$T(s) = \frac{s^2 + 2(10)^6}{1000s}$$

Replacing s with $T(s)$ in the right-hand side of Eq. (7.58) yields the final bandpass transfer function

$$H(s) = \frac{9.826(10)^5 s^2}{s^4 + 1097.7s^3 + 5.1025(10)^6 s^2 + 2.195(10)^9 s + 4(10)^{12}}$$

The amplitude response $|H(j\omega)|$ of this filter is shown in Fig. 7.30a. ■

We may use a similar procedure for the Butterworth filter. Compared to Chebyshev design, Butterworth filter design involves two additional steps. First, we need to compute the cutoff frequency ω_c of the prototype filter. For a Chebyshev filter, the critical frequency happens to the frequency where the gain is G_p. This frequency is $\omega = 1$ in the prototype filter. For Butterworth, on the other hand, the critical frequency is the half power (or 3 dB-cutoff) frequency ω_c, which is not necessarily the frequency where the gain is G_p. To find the transfer function of the Butterworth prototype filter, it is essential to know ω_c. Once we know ω_c, the prototype filter transfer function is obtained by replacing s with s/ω_c in the normalized transfer function $\mathcal{H}(s)$. This step is also unnecessary in the Chebyshev filter design. We shall demonstrate the procedure for the Butterworth filter design by an example below.

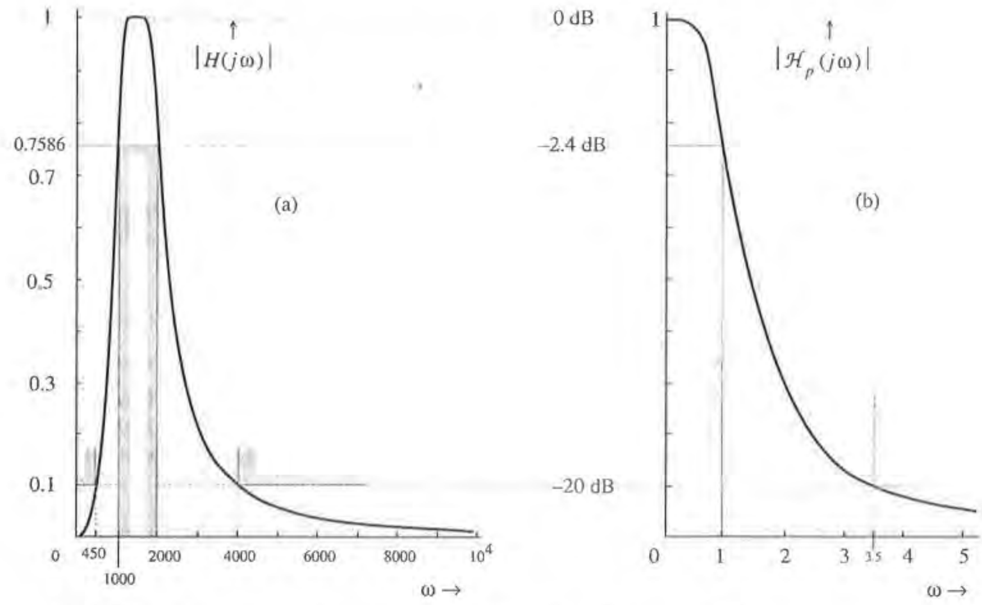


Fig. 7.31 Butterworth Bandpass Filter Design for Example 7.10.

Example 7.10

Design a Butterworth bandpass filter with the amplitude response specifications illustrated in Fig. 7.31a with $\omega_{p_1} = 1000$, $\omega_{p_2} = 2000$, $\omega_{s_1} = 450$, $\omega_{s_2} = 4000$, $G_p = 0.7586\,(-2.4\,\text{dB})$, and $G_s = 0.1\,(-20\,\text{dB})$.

As in the previous example, the solution is executed in two steps: in the first step, we determine the lowpass prototype filter transfer function $\mathcal{H}_p(s)$. In the second step, the desired bandpass filter transfer function is obtained from $\mathcal{H}_p(s)$ by substituting s with $T(s)$, the lowpass to bandpass transformation in Eq. (7.57).

Step 1: Find $\mathcal{H}_p(s)$, the lowpass prototype filter transfer function.

This goal is accomplished in 5 substeps used in the design of the lowpass Butterworth filter (see Example 7.6):

Step 1.1: Find ω_s for the prototype filter.

For the prototype lowpass filter transfer function $\mathcal{H}_p(s)$ with the amplitude response shown in Fig. 7.31b, the frequency ω_s is found [using Eq. (7.56)] to be the smaller of

$$\frac{(1000)(2000) - (450)^2}{450(2000 - 1000)} = 3.99 \quad \text{and} \quad \frac{(4000)^2 - (1000)(2000)}{4000(2000 - 1000)} = 3.5$$

which is 3.5, as depicted in Fig. 7.31b.

Step 1.2: Determine n

For a prototype lowpass filter in Fig. 7.29b, $\hat{G}_p = -2.4$ dB, $\hat{G}_s = -20$ dB, $\omega_p = 1$, and $\omega_s = 3.5$. Hence, according to Eq. (7.39), the Butterworth filter order n required to meet these specifications is

$$n = \frac{1}{2\,\log 3.5} \log\left[\frac{10^2 - 1}{10^{0.24} - 1}\right] = 1.955$$

which is rounded up to $n = 2$.

Step 1.3: Determine ω_c
In this step (which is not necessary for the Chebyshev design), we determine the 3 dB cutoff frequency ω_c for the prototype filter. Use of Eq. (7.41) yields

$$\omega_c = \frac{3.5}{(10^2-1)^{1/4}} = 1.10958$$

Step 1.4: Determine the normalized transfer function $\mathcal{H}(s)$
The normalized second-order lowpass Butterworth transfer function from Table 7.1 is

$$\mathcal{H}(s) = \frac{1}{s^2+\sqrt{2}s+1}$$

This is the transfer function of a normalized filter (meaning that $\omega_c = 1$).

Step 1.5: Determine the prototype filter transfer function $\mathcal{H}_p(s)$
The prototype filter transfer function $\mathcal{H}_p(s)$ is obtained by substituting s with $s/\omega_c = s/1.10958$ in the normalized transfer function $\mathcal{H}(s)$ found in step 1.4 as

$$\mathcal{H}_p(s) = \frac{(1.10958)^2}{s^2+\sqrt{2}(1.10958)s+(1.10958)^2} = \frac{1.231}{s^2+1.5692s+1.2312} \tag{7.59}$$

The amplitude response of this prototype filter is illustrated in Fig. 7.31b.

Step 2: Find the desired bandpass filter transfer function $H(s)$ using the lowpass to bandpass transformation.

Finally the desired bandpass filter transfer function $H(s)$ is obtained from $\mathcal{H}_p(s)$ by replacing s with $T(s)$, where [see Eq. (7.57)]

$$T(s) = \frac{s^2+2(10)^6}{1000s}$$

Replacing s with $T(s)$ in the right-hand side of Eq. (7.59) yields the final bandpass transfer function

$$H(s) = \frac{1.2312(10)^6\,s^2}{s^4+1569\,s^3+5.2312(10)^6\,s^2+3.1384(10)^9 s+4(10)^{12}}$$

The amplitude response $|H(j\omega)|$ of this filter is shown in Fig. 7.31a. ∎

⊙ **Computer Example C7.12**

Design a bandpass filter for the specifications in Example 7.10 using functions from *Signal Processing Toolbox* in MATLAB. We shall give here MATLAB functions for the four types of filters.

For bandpass filters, we use the same functions as those used for lowpass filter in Examples C7.6, C7.8-C7.10, with one difference: Wp and Ws are 2 element vectors as Wp=[Wp1 Wp2], Ws=[Ws1 Ws2].

```
Wp=[1000 2000];Ws=[450 4000];Gp=-2.4;Gs=-20;
% Butterworth
[n,Wn]=buttord(Wp,Ws,-Gp,-Gs,'s')
[num,den]=butter(n,Wn,'s')
% Chebyshev
[n,Wn]=cheb1ord(Wp,Ws,-Gp,-Gs,'s');
[num,den]=cheby1(N,-Gp,Wn,'s')
```

```
% Inverse Chebyshev
[n,Ws]=cheb2ord(Wp,Ws,-Gp,-Gs,'s');
[num,den]=cheby2(n,-Gs,Ws,'s')
% Elliptic filter
[n,Wn]=ellipord(Wp,Ws,-Gp,-Gs,'s');
[num,den]=ellip(n,-Gp,-Gs,Wn,'s')
```

To plot amplitude response, we can use the last three functions from Example C7.5.

7.7-3 Bandstop Filters

Figure 7.32a shows an amplitude response of a typical bandstop filter. To design such a filter, we first find $\mathcal{H}_p(s)$, the transfer function of a prototype lowpass filter, to meet the specifications in Fig. 7.32b, where ω_s is given by the smaller of

$$\frac{(\omega_{p_2}-\omega_{p_1})\omega_{s_1}}{\omega_{p_1}\omega_{p_2}-{\omega_{s_1}}^2} \quad \text{or} \quad \frac{(\omega_{p_2}-\omega_{p_1})\omega_{s_2}}{{\omega_{s_2}}^2-\omega_{p_1}\omega_{p_2}} \tag{7.60}$$

The desired transfer function of the bandstop filter to satisfy the specifications in Fig. 7.32a is obtained from $\mathcal{H}_p(s)$ by replacing s with $T(s)$, where

$$T(s)=\frac{(\omega_{p_2}-\omega_{p_1})s}{s^2+\omega_{p_1}\omega_{p_2}} \tag{7.61}$$

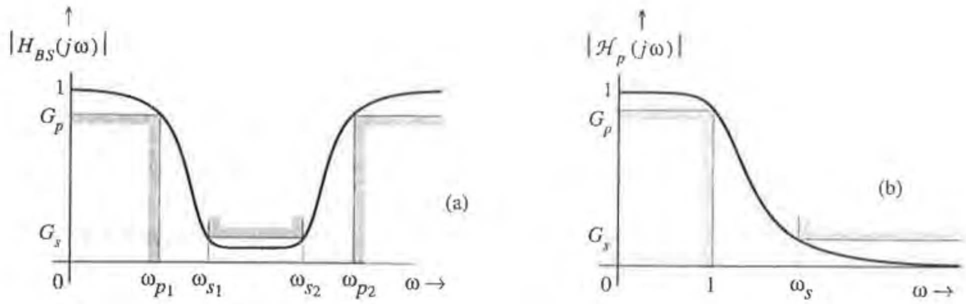


Fig. 7.32 Frequency transformation for bandstop filters.

■ **Example 7.11**

Design a Butterworth bandstop filter with the specifications depicted in Fig. 7.33a with $\omega_{p_1}=60$, $\omega_{p_2}=260$, $\omega_{s_1}=100$, $\omega_{s_2}=150$, $G_p=0.776\,(-2.2\,\text{dB})$, and $G_s=0.1\,(-20\,\text{dB})$.

In the first step we shall determine the prototype lowpass filter transfer function $\mathcal{H}_p(s)$, and in the second step we use the lowpass to bandstop transformation in Eq. (7.61) to obtain the desired bandstop filter transfer function $H(s)$.

Step 1: Find $\mathcal{H}_p(s)$, the lowpass prototype filter transfer function.

This goal is accomplished in 5 substeps used in the design of the lowpass Butterworth filter (see Example 7.6):

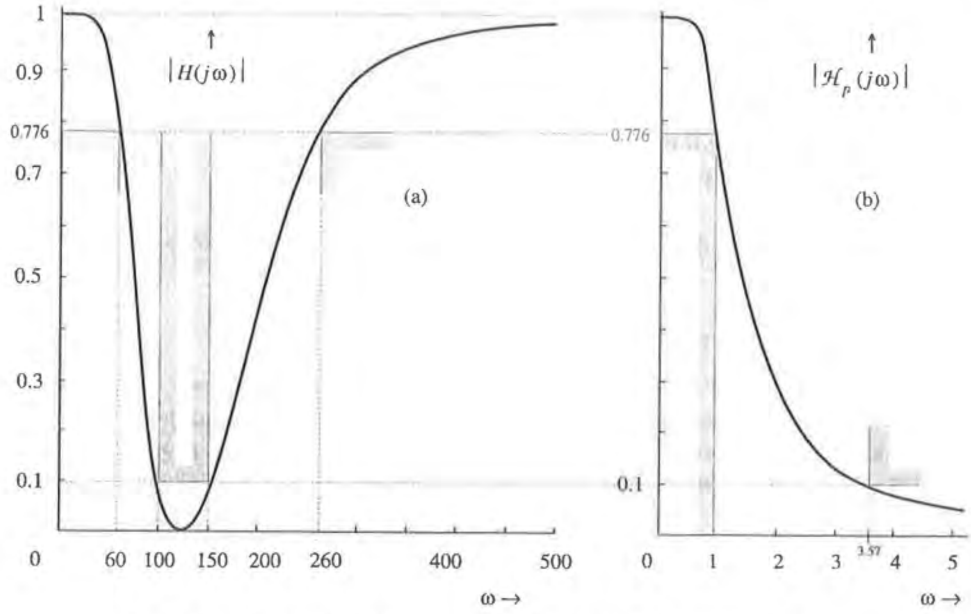


Fig. 7.33 Butterworth Bandstop Filter Design for Example 7.11.

Step 1.1: Find ω_s for the prototype filter.
For the prototype lowpass filter transfer function $\mathcal{H}_p(s)$ with the specifications illustrated in Fig. 7.32b, the frequency ω_s is found [using Eq. (7.60)] to be the smaller of

$$\frac{(100)(260-60)}{(260)(60)-100^2}=3.57 \quad \text{and} \quad \frac{150(260-60)}{150^2-(260)(60)}=4.347$$

which is 3.57, as shown in Fig. 7.33b.

Step 1.2: Determine n
For the prototype lowpass filter in Fig. 7.33b, $\hat{G}_p=-2.2$ dB, $\hat{G}_s=-20$ dB, $\omega_p=1$, and $\omega_s=3.57$. According to Eq. (7.39), the Butterworth filter order n required to meet these specifications is

$$n=\frac{1}{2\,\log(3.57)}\left[\frac{10^2-1}{10^{0.22}-1}\right]=1.9689$$

We round up the value of n to 2.

Step 1.3: Determine ω_c
The half power frequency ω_c for the prototype Butterworth filter, using Eq. (7.40) with $\omega_p=1$, is

$$\omega_c=\frac{\omega_p}{(10^{-\hat{G}_p/10}-1)^{\frac{1}{4}}}=\frac{1}{(10^{0.22}-1)^{\frac{1}{4}}}=1.1096$$

Step 1.4: Determine the normalized transfer function
The transfer function of the second-order normalized Butterworth filter from the Table 7.1 is

$$\mathcal{H}(s)=\frac{1}{s^2+\sqrt{2}s+1} \tag{7.62}$$

Step 1.5: Determine the prototype filter transfer function $\mathcal{H}_p(s)$
The prototype filter transfer function $\mathcal{H}_p(s)$ is obtained by substituting s with $s/\omega_c = s/1.1096$ in the normalized transfer function $\mathcal{H}(s)$ in step 1.4. This move yields

$$\mathcal{H}_p(s) = \frac{1}{(\frac{s}{\omega_c})^2 + \sqrt{2}\frac{s}{\omega_c} + 1} = \frac{1.2312}{s^2 + 1.5692s + 1.2312} \tag{7.63}$$

The amplitude response of this prototype filter is depicted in Fig. 7.33b.

Step 2: Find the desired bandstop filter transfer function $H(s)$ using the lowpass to bandstop transformation

Finally, the desired transfer function $H(s)$ of the bandpass filter with specifications illustrated in Fig. 7.33a is obtained from $\mathcal{H}_p(s)$ by replacing s with $T(s)$, where [see Eq. (7.61)]

$$T(s) = \frac{200s}{s^2 + 15,600}$$

Replacing s with $T(s)$ in the right-hand side of Eq. (7.63) yields the final bandstop transfer function

$$\begin{aligned} H(s) &= \frac{1.2312}{(\frac{200s}{s^2+15,600})^2 + 1.5692(\frac{200s}{s^2+15,600}) + 1.2312} \\ &= \frac{(s^2+15600)^2}{s^4 + 254.9s^3 + 63690.9s^2 + (3.977)10^6 s + (2.433)10^8} \end{aligned}$$

The amplitude response $|H(j\omega)|$ is shown in Fig. 7.33a. ■

⊙ **Computer Example C7.13**

Design the bandstop filter for the specifications in Example 7.11 using functions from *Signal Processing Toolbox* in MATLAB. We shall give here MATLAB functions for all the four types of filters.

```
Wp=[60 260]; Ws=[100 150]; Gp=-2.2;Gs=-20;
% Butterworth
[n,Wn]=buttord(Wp,Ws,-Gp,-Gs,'s')
[num,den]=butter(n,Wn,'stop','s')
% Chebyshev
[n,Wn]=cheb1ord(Wp,Ws,-Gp,-Gs,'s')
[num,den]=cheby1(n,-Gp,Wn,'stop','s')
% Inverse Chebyshev
[n,Wn]=cheb2ord(Wp,Ws,-Gp,-Gs,'s')
[num,den]=cheby2(n,-Gs,Wn,'stop','s')
% Elliptic
[n,Wn]=ellipord(Wp,Ws,-Gp,-Gs,'s')
[num,den]=ellip(n,-Gp,-Gs,Wn,'stop','s')
```
⊙

7.8 Filters to Satisfy Distortionless transmission Conditions

The purpose of a filter is to suppress unwanted frequency components and to transmit the desired frequency components without distortion. In Sec. 4.4, we saw that this requires the filter amplitude response to be constant and the phase response to be a linear function of ω over the passband.

The filters discussed so far have stressed the constancy of the amplitude response. The linearity of the phase response has been ignored. As we saw earlier, the human ear is sensitive to amplitude distortion but somewhat insensitive to phase distortion. For this reason filters in audio application are designed primarily for constant amplitude response, and the phase response is only a secondary consideration.

We also saw earlier that the human eye is sensitive to phase distortion and relatively insensitive to amplitude distortion. Therefore, in video applications we cannot ignore phase distortion. In pulse communication, both the amplitude and the phase distortion are important for correct information transmission. Thus, in practice, we also need to design filters primarily for phase linearity in video applications. In pulse communication applications, it is important to have filters with constant amplitude response and a linear phase response. We shall briefly discuss some aspects and approaches to the design of such filters. More discussion appears in the literature.[2]

We showed [see Eq. (4.59)] that the time delay t_d resulting from the signal transmission through a filter is the negative of the slope of the filter phase response $\angle H(j\omega)$; that is,

$$t_d(\omega) = -\frac{d}{d\omega}\angle H(j\omega) \tag{7.64}$$

If the slope of $\angle H(j\omega)$ is constant over the desired band (that is, if $\angle H(j\omega)$ is linear with ω), all the components are delayed by the same time interval t_d. In this case the output is a replica of the input, assuming that all components are attenuated equally; that is, $|H(j\omega)| =$ constant over the passband.

If the slope of the phase response is not constant, t_d, the time delay, varies with frequency. This variation means that different frequency components undergo different amounts of time delay, and consequently the output waveform will not be a replica of the input waveform even if the amplitude response is constant over the passband. A good way of judging phase linearity is to plot t_d as a function of frequency. For a distortionless system, t_d (the negative slope of $\angle H(j\omega)$) should be constant over the band of interest. This is in addition to the requirement of constancy of the amplitude response.

Generally speaking, the two requirements of distortionless transmission conflict. The more we approach the ideal amplitude response, the further we deviate from the ideal phase response. The sharper the cutoff characteristic (smaller the transition band), the more nonlinear is the phase response near the transition band. We can verify this fact from Fig. 7.34, which shows the delay characteristic of the Butterworth and the Chebyshev family of filters. The Chebyshev filter, which has a sharper cutoff than that of the Butterworth, shows considerably more variation in time delay of various frequency components as compared to that of the Butterworth.

For the applications where the phase linearity is also important, there are two possible approaches:

1 If $t_d =$ constant (phase linearity) is the primary requirement, we design a filter for which t_d is maximally flat around $\omega = 0$ and accept the resulting amplitude response, which may not be so flat nor have a sharp cutoff characteristic.

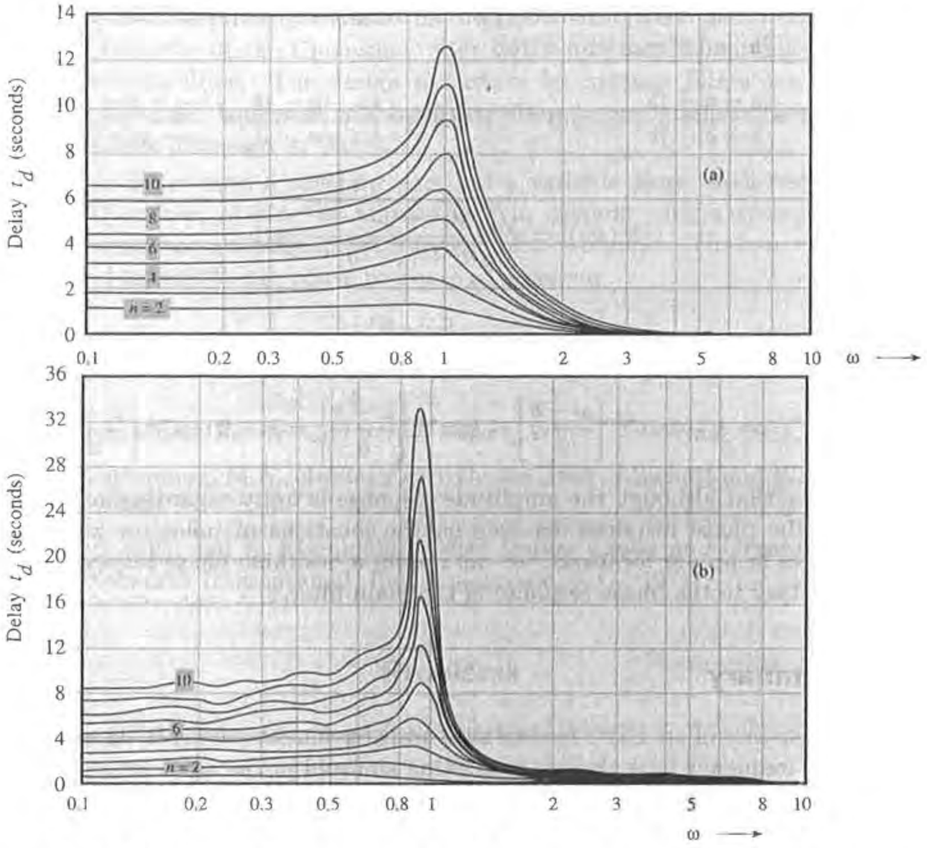


Fig. 7.34 Delay characteristics of the Butterworth and Chebyshev filters.

Contrast this with the Butterworth filter, which is designed to yield the maximally flat amplitude response at $\omega = 0$ without any attention to the phase response. A family of filters which yields a maximally flat t_d goes under the name **Bessel-Thomson** filters, which uses the nth-order Bessel polynomial in the denominator of nth-order $H(s)$.

2 If both amplitude and phase response are important, we start with a filter to satisfy the amplitude response specifications, disregarding the phase response specifications. We cascade this filter with another filter, an **equalizer**, whose amplitude response is flat for all frequencies **(the allpass filter)** and whose t_d characteristic is complementary to that of the main filter in such a way that their composite phase characteristic is approximately linear. The cascade thus has linear phase and the amplitude response of the main filter (as required).

Allpass Filters

An allpass filter has equal number of poles and zeros. All the poles are in the LHP (left half plane) for stability. All the zeros are mirror images of the poles about the imaginary axis. In other words, for every pole at $-a + jb$, there is a zero at $a + jb$. Thus, all the zeros are in the RHP. Any filter with this kind of pole-zero configuration is an allpass filter; that is, its amplitude response is constant for all frequencies. We can verify this assertion by considering a transfer function with a pole at $-a + jb$ and a zero at $a + jb$:

$$H(s) = \frac{s - a - jb}{s + a - jb} \qquad \text{and} \qquad H(j\omega) = \frac{j\omega - a - jb}{j\omega + a - jb} = \frac{-a + j(\omega - b)}{a + j(\omega - b)}$$

Therefore

$$|H(j\omega)| = \frac{\sqrt{(-a)^2 + (\omega - b)^2}}{\sqrt{a^2 + (\omega - b)^2}} = 1 \tag{7.65}$$

$$\begin{aligned}\angle H(j\omega) &= \tan^{-1}\left[\frac{\omega - b}{-a}\right] - \tan^{-1}\left[\frac{\omega - b}{a}\right] \\ &= \pi - \tan^{-1}\left[\frac{\omega - b}{a}\right] - \tan^{-1}\left[\frac{\omega - b}{a}\right] = \pi - 2\tan^{-1}\left[\frac{\omega - b}{a}\right]\end{aligned} \tag{7.66}$$

Observe that although the amplitude response is unity regardless of pole-zero locations, the phase response depends on the locations of poles (or zeros). By placing poles in proper locations, we can obtain a desirable phase response that is complementary to the phase response of the main filter.

7.9 Summary

The response of an LTIC system with transfer function $H(s)$ to an everlasting sinusoid of frequency ω is also an everlasting sinusoid of the same frequency. The output amplitude is $|H(j\omega)|$ times the input amplitude, and the output sinusoid is shifted in phase with respect to the input sinusoid by $\angle H(j\omega)$ radians. The plot of $|H(j\omega)|$ vs ω indicates the amplitude gain of sinusoids of various frequencies and is called the *amplitude response* of the system. The plot of $\angle H(j\omega)$ vs ω indicates the phase shift of sinusoids of various frequencies and is called the phase response.

Plotting of the frequency response is remarkably simplified by using logarithmic units for amplitude as well as frequency. Such plots are known as the Bode plots. The use of logarithmic units makes it possible to add (rather than multiply) the amplitude response of four basic types of factors that occur in transfer functions: (1) a constant (2) a pole or a zero at the origin (3) a first order pole or a zero, and (4) complex conjugate poles or zeros. For phase plots, we use linear units for phase and logarithmic units for the frequency. The phases corresponding to the three basic types of factors mentioned above add. The asymptotic properties of the amplitude and phase responses allow their plotting with remarkable ease even for transfer functions of high orders.

The frequency response of a system is determined by the locations in the complex plane of poles and zeros of its transfer function. We can design frequency selective filters by proper placement of its transfer function poles and zeros. Placing a pole (a zero) near a frequency $j\omega_0$ in the complex plane enhances (suppresses) the frequency response at the frequency $\omega = \omega_0$. With this concept, a proper combination of poles and zeros at suitable locations can yield desired filter characteristics.

Two families of analog filters are considered: Butterworth and Chebyshev. The Butterworth filter has a maximally flat amplitude response over the passband.

The Chebyshev amplitude response has ripples in the passband. On the other hand, the behavior of the Chebyshev filter in the stopband is superior to that of the Butterworth filter. The design procedure for lowpass filters can be readily applied to highpass, bandpass, and bandstop filters by using appropriate frequency transformations discussed in Sec. 7.7.

Allpass filters have a constant gain but a variable phase with respect to frequency. Therefore, placing an allpass filter in cascade with a system leaves its amplitude response unchanged, but alters its phase response. Thus, an allpass filter can be used to modify the phase response of a system.

References

1. Wai-Kai Chen, *Passive and active Filters*, Wiley, New York, 1986.
2. Van Valkenberg, M.E., *Analog Filter Design*, Holt, Rinehart and Winston, New York, 1982.
3. Christian E., and E. Eisenmann, *Filter Design Tables and Graphs*, Transmission Networks International, Inc., Knightdale, N.C., 1977.

Problems

7.1-1 For an LTIC system described by the transfer function

$$H(s) = \frac{s+2}{s^2+5s+4}$$

find the response to the following everlasting sinusoidal inputs: **(a)** $5\cos(2t+30°)$ **(b)** $10\sin(2t+45°)$ **(c)** $10\cos(3t+40°)$. Observe that these are everlasting sinusoids.

7.1-2 For an LTIC system described by the transfer function

$$H(s) = \frac{s+3}{(s+2)^2}$$

find the steady-state system response to the following inputs:
(a) $10u(t)$ **(b)** $\cos(2t+60°)u(t)$ **(c)** $\sin(3t-45°)u(t)$ **(d)** $e^{j3t}u(t)$

7.1-3 For an allpass filter specified by the transfer function

$$H(s) = \frac{-(s-10)}{s+10}$$

find the system response to the following (everlasting) inputs: **(a)** $e^{j\omega t}$ **(b)** $\cos(\omega t+\theta)$ **(c)** $\cos t$ **(d)** $\sin 2t$ **(e)** $\cos 10t$ **(f)** $\cos 100t$.
Comment on the filter response.

7.2-1 Sketch Bode plots for the following transfer functions:

$$\text{(a)}\ \frac{s(s+100)}{(s+2)(s+20)} \quad \text{(b)}\ \frac{(s+10)(s+20)}{s^2(s+100)} \quad \text{(c)}\ \frac{(s+10)(s+200)}{(s+20)^2(s+1000)}$$

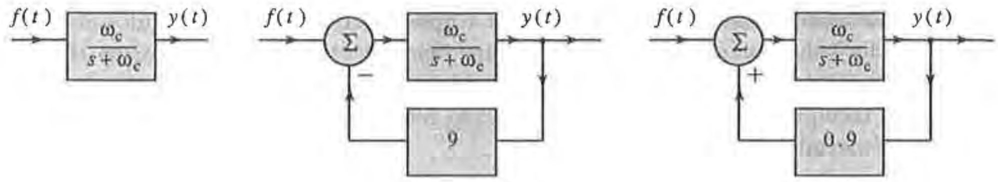


Fig. P7.3-1

7.2-2 Repeat Prob. 7.2-1 if

$$\text{(a)}\ \frac{s^2}{(s+1)(s^2+4s+16)} \quad \text{(b)}\ \frac{s}{(s+1)(s^2+14.14s+100)} \quad \text{(c)}\ \frac{(s+10)}{s(s^2+14.14s+100)}$$

7.3-1 Feedback can be used to increase (or decrease) the system bandwidth. Consider a system in Fig. P7.3-1a with transfer function $G(s) = \frac{\omega_c}{s+\omega_c}$.
(a) Show that the 3 dB bandwidth of this system is ω_c.
(b) To increase the bandwidth of this system, we use negative feedback with $H(s) = 9$, as depicted in Fig. P7.3-1b. Show that the 3 dB bandwidth of this system is $10\omega_c$.
(c) To decrease the bandwidth of this system, we use positive feedback with $H(s) = -0.9$, as illustrated in Fig. P7.3-1c. Show that the 3 dB bandwidth of this system is $\omega_c/10$.
(d) The system gain at dc times its 3 dB bandwidth is called the **gain-bandwidth product** of a system. Show that this product is the same for all the three systems in Fig. P7.3-1. This result shows that if we increase the bandwidth, the gain decreases and vice versa.

7.4-1 Using the graphical method of Sec 7.4-1, draw a rough sketch of the amplitude and phase response of an LTIC system described by the transfer function

$$H(s) = \frac{s^2-2s+50}{s^2+2s+50} = \frac{(s-1-j7)(s-1+j7)}{(s+1-j7)(s+1+j7)}$$

What kind of filter is this?

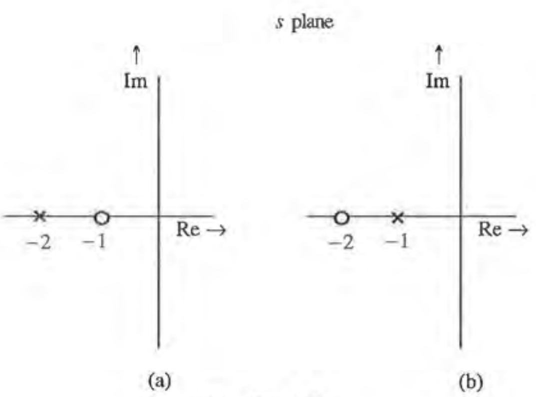


Fig. P7.4-2

7.4-2 Using the graphical method of Sec. 7.4-1, draw a rough sketch of the amplitude and phase response of LTIC systems whose pole-zero plots are shown in Fig. P7.4-2.

7.4-3 Design a second-order bandpass filter with center frequency $\omega = 10$. The gain should be zero at $\omega = 0$ and at $\omega = \infty$. Select poles at $-a \pm j10$. Leave your answer in terms of a. Explain the influence of a on the frequency response.

7.5-1 Determine the transfer function $H(s)$ and the amplitude response $H(j\omega)$ for a third-order lowpass Butterworth filter if the 3 dB cutoff frequency $\omega_c = 100$. Find your answer without using Tables 7.1 or 7.2. Verify your answer using either of these Tables.

7.5-2 Determine n, the order of a lowpass Butterworth filter, and the corresponding cutoff frequency ω_c required to satisfy the following lowpass filter specifications. Find both the values of ω_c, the one that oversatisfies the passband specifications, and the one that oversatisfies the stopband specifications.

(a) $\hat{G}_p \geq -0.5$ dB, $\hat{G}_s \leq -20$ dB, $\omega_p = 100$ rad/s, and $\omega_s = 200$ rad/s.
(b) $G_p \geq 0.9885$, $G_s \leq 10^{-3}$, $\omega_p = 1000$ rad/s, and $\omega_s = 2000$ rad/s.
(c) The gain at $3\omega_c$ is required to be no greater than -50 dB.

7.5-3 Find the transfer function $H(s)$ and the amplitude response $H(j\omega)$ for a lowpass Butterworth filter to satisfy the specifications: $\hat{G}_p \geq -3$ dB, $\hat{G}_s \leq -14$ dB, $\omega_p = 100,000$ rad/s, and $\omega_s = 150,000$ rad/s. It is desirable to oversatisfy (if possible) the requirement of $\hat{G}_s$. Determine the $\hat{G}_p$ and $\hat{G}_s$ of your design.

7.6-1 Repeat Prob. 7.5-1 for a Chebyshev filter. Do not use Tables.

7.6-2 Design a lowpass Chebyshev filter to satisfy the specifications: $\hat{G}_p \geq -1$ dB, $\hat{G}_s \leq -22$ dB, $\omega_p = 100$ rad/s, and $\omega_s = 200$ rad/s.

7.6-3 Design a lowpass Chebyshev filter to satisfy the specifications: $\hat{G}_p \geq -2$ dB, $\hat{G}_s \leq -25$ dB, $\omega_p = 10$ rad/s, and $\omega_s = 15$ rad/s.

7.6-4 Design a lowpass Chebyshev filter whose 3 dB cutoff frequency is ω_c, and the gain drops to -50 dB at $3\omega_c$.

7.7-1 Find the transfer function $H(s)$ for a highpass Butterworth filter to satisfy the specifications: $\hat{G}_s \leq -20$ dB, $\hat{G}_p \geq -1$ dB, $\omega_s = 10$, and $\omega_p = 20$.

7.7-2 Find the transfer function $H(s)$ for a highpass Chebyshev filter to satisfy the specifications: $\hat{G}_s \leq -22$ dB, $\hat{G}_p \geq -1$ dB, $\omega_s = 10$, and $\omega_p = 20$

7.7-3 Find the transfer function $H(s)$ for a Butterworth bandpass filter to satisfy the specifications: $\hat{G}_s \leq -17$ dB, $\hat{G}_p \geq -3$ dB, $\omega_{p_1} = 100$ rad/s, $\omega_{p_2} = 250$ rad/s, and $\omega_{s_1} = 40$ rad/s, $\omega_{s_2} = 500$ rad/s.

7.7-4 Find the transfer function $H(s)$ for a Chebyshev bandpass filter to satisfy the specifications: $\hat{G}_s \leq -17$ dB, $\hat{r} \leq 1$ dB, $\omega_{p_1} = 100$ rad/s, $\omega_{p_2} = 250$ rad/s, and $\omega_{s_1} = 40$ rad/s, $\omega_{s_2} = 500$ rad/s.

7.7-5 Find the transfer function $H(s)$ for a Butterworth bandstop filter to satisfy the specifications: $\hat{G}_s \leq -24$ dB, $\hat{G}_p \geq -3$ dB, $\omega_{p_1} = 20$ rad/s, $\omega_{p_2} = 60$ rad/s, and $\omega_{s_1} = 30$ rad/s, $\omega_{s_2} = 38$ rad/s.