

Frequency Response and Digital Filters

Filtering characteristics of a system are specified by its frequency response. For this reason it is important to study frequency response of discrete-time systems, which is very similar to the frequency response of continuous-time systems with some significant differences.

12.1 Frequency Response of Discrete-Time Systems

For (asymptotically stable) continuous-time systems we showed that the system response to an input $e^{j\omega t}$ is $H(j\omega)e^{j\omega t}$, and that the response to an input $\cos \omega t$ is $|H(j\omega)| \cos[\omega t + \angle H(j\omega)]$. Similar results hold for discrete-time systems. We now show that for an (asymptotically stable) LTID system, the system response to an input $e^{j\Omega k}$ is $H[e^{j\Omega}]e^{j\Omega k}$ and the response to an input $\cos \Omega k$ is $|H[e^{j\Omega}]| \cos(\Omega k + \angle H[e^{j\Omega}])$.

The proof is similar to the one used in continuous-time systems. In Sec. 9.4-2 we showed that an LTID system response to an (everlasting) exponential z^k is also an (everlasting) exponential $H[z]z^k$. It is helpful to represent this relationship by a directed arrow notation as

$$z^k \implies H[z]z^k \quad (12.1)$$

Setting $z = e^{\pm j\Omega}$ in this relationship yields

$$e^{j\Omega k} \implies H[e^{j\Omega}]e^{j\Omega k} \quad (12.2a)$$

$$e^{-j\Omega k} \implies H[e^{-j\Omega}]e^{-j\Omega k} \quad (12.2b)$$

Addition of these two equations yields

$$2 \cos \Omega k \implies H[e^{j\Omega}]e^{j\Omega k} + H[e^{-j\Omega}]e^{-j\Omega k} = 2\text{Re} \left(H[e^{j\Omega}]e^{j\Omega k} \right) \quad (12.3)$$

Expressing $H[e^{j\Omega}]$ in the polar form

$$H[e^{j\Omega}] = |H[e^{j\Omega}]|e^{j\angle H[e^{j\Omega}]} \quad (12.4)$$

Eq. (12.3) can be expressed as

$$\cos \Omega k \implies |H[e^{j\Omega}]| \cos \left(\Omega k + \angle H[e^{j\Omega}] \right) \quad (12.5)$$

In other words, the system response $y[k]$ to a sinusoidal input $\cos \Omega k$ is given by

$$y[k] = |H[e^{j\Omega}]| \cos \left(\Omega k + \angle H[e^{j\Omega}] \right) \quad (12.6a)$$

Following the same argument, the system response to a sinusoid $\cos(\Omega k + \theta)$ is

$$y[k] = |H[e^{j\Omega}]| \cos \left(\Omega k + \theta + \angle H[e^{j\Omega}] \right) \quad (12.6b)$$

This result applies only to asymptotically stable systems because Eq. (12.1) is valid only for values of z lying in the region of convergence of $H[z]$. For $z = e^{j\Omega}$, z lies on the unit circle ($|z| = 1$). The region of convergence for unstable and marginally stable systems does not include the unit circle.

This important result shows that the response of an asymptotically stable LTID system to a discrete-time sinusoidal input of frequency Ω is also a discrete-time sinusoid of the same frequency. *The amplitude of the output sinusoid is $|H[e^{j\Omega}]|$ times the input amplitude, and the phase of the output sinusoid is shifted by $\angle H[e^{j\Omega}]$ with respect to the input phase.* Clearly $|H[e^{j\Omega}]|$ is the amplitude gain, and a plot of $|H[e^{j\Omega}]|$ versus Ω is the amplitude response of the discrete-time system. Similarly, $\angle H[e^{j\Omega}]$ is the phase response of the system, and a plot of $\angle H[e^{j\Omega}]$ vs Ω shows how the system modifies or shifts the phase of the input sinusoid. Note that $H[e^{j\Omega}]$ incorporates the information of both amplitude and phase response and therefore is called the **frequency response** of the system.

These results, although parallel to those for continuous-time systems, differ from them in one significant aspect. In the continuous-time case, the frequency response is $H(j\omega)$. A parallel result for the discrete-time case would lead to frequency response $H[j\Omega]$. Instead, we found the frequency response to be $H[e^{j\Omega}]$. This deviation causes some interesting differences between the behavior of continuous-time and discrete-time systems.

Steady-State Response to Causal Sinusoidal Input

As in the case of continuous-time systems, we can show that the response of an LTID system to a causal sinusoidal input $\cos \Omega k u[k]$ is $y[k]$ in Eq. (12.6a), plus a natural component consisting of the characteristic modes (see Prob. 12.1-4). For a stable system, all the modes decay exponentially, and only the sinusoidal component in Eq. (12.6a) persists. For this reason, this component is called the sinusoidal *steady-state* response of the system. Thus, $y_{ss}[k]$, the steady-state response of a system to a causal sinusoidal input $\cos \Omega k u[k]$, is

$$y_{ss}[k] = |H[e^{j\Omega}]| \cos \left(\Omega k + \angle H[e^{j\Omega}] \right)$$

System Response to Sampled Continuous-Time Sinusoids

So far we have considered the system response of a discrete-time system to a discrete-time sinusoid $\cos \Omega k$ (or exponential $e^{j\Omega k}$). In practice, the input may be a

sampled continuous-time sinusoid $\cos \omega t$ (or an exponential $e^{j\omega t}$). When a sinusoid $\cos \omega t$ is sampled with sampling interval T , the resulting signal is a discrete-time sinusoid $\cos \omega kT$. Therefore, all the results developed in this section apply if we substitute ωT for Ω :

$$\Omega = \omega T$$

■ **Example 12.1**

For a system specified by the equation

$$y[k+1] - 0.8y[k] = f[k+1]$$

find the system response to the input (a) $1^k = 1$ (b) $\cos[\frac{\pi}{6}k - 0.2]$

(c) a sampled sinusoid $\cos 1500t$ with sampling interval $T = 0.001$.

The system equation can be expressed as

$$(E - 0.8)y[k] = Ef[k]$$

Therefore, the transfer function of the system is

$$H[z] = \frac{z}{z - 0.8} = \frac{1}{1 - 0.8z^{-1}}$$

The frequency response is

$$\begin{aligned} H[e^{j\Omega}] &= \frac{1}{1 - 0.8e^{-j\Omega}} & (12.7) \\ &= \frac{1}{1 - 0.8(\cos \Omega - j \sin \Omega)} \\ &= \frac{1}{(1 - 0.8 \cos \Omega) + j0.8 \sin \Omega} \end{aligned}$$

Therefore

$$\begin{aligned} |H[e^{j\Omega}]| &= \frac{1}{\sqrt{(1 - 0.8 \cos \Omega)^2 + (0.8 \sin \Omega)^2}} \\ &= \frac{1}{\sqrt{1.64 - 1.6 \cos \Omega}} \end{aligned} \quad (12.8a)$$

and

$$\angle H[e^{j\Omega}] = -\tan^{-1} \left[\frac{0.8 \sin \Omega}{1 - 0.8 \cos \Omega} \right] \quad (12.8b)$$

The amplitude response $|H[e^{j\Omega}]|$ can also be obtained by observing that $|H|^2 = HH^*$. Therefore

$$\begin{aligned} |H[e^{j\Omega}]|^2 &= H[e^{j\Omega}]H^*[e^{j\Omega}] \\ &= H[e^{j\Omega}]H[e^{-j\Omega}] \end{aligned} \quad (12.9)$$

From Eq. (12.7) it follows that

$$\begin{aligned} |H[e^{j\Omega}]|^2 &= \left(\frac{1}{1 - 0.8e^{-j\Omega}} \right) \left(\frac{1}{1 - 0.8e^{j\Omega}} \right) \\ &= \frac{1}{1.64 - 1.6 \cos \Omega} \end{aligned}$$

which yields the result found earlier in Eq. (12.8a).

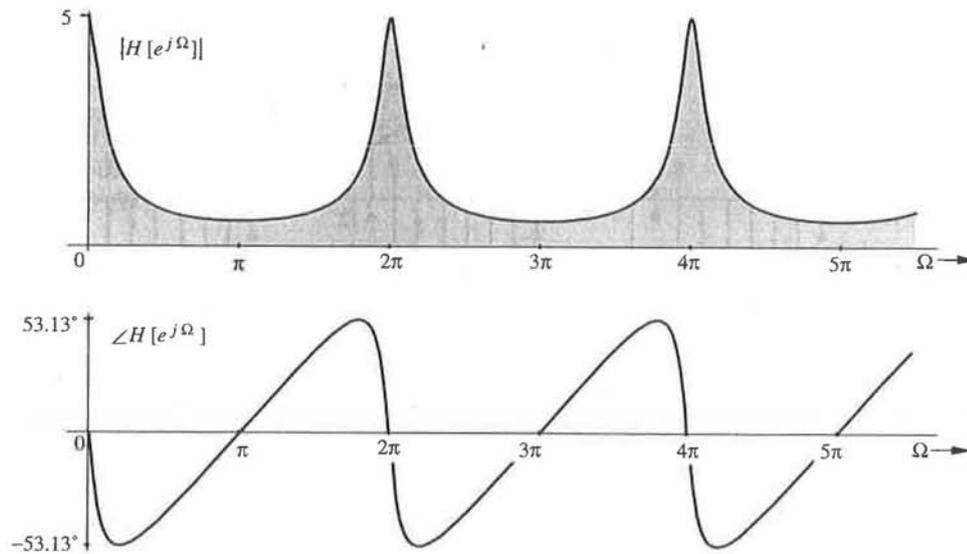


Fig. 12.1 Frequency response of an LTID system in Example 12.1.

Figure 12.1 shows plots of amplitude and phase response as functions of Ω . We now compute the amplitude and the phase response for the various inputs:

(a) $f[k] = 1^k = 1$

Since $1^k = (e^{j\Omega})^k$ with $\Omega = 0$, the amplitude response is $H[e^{j0}]$. From Eq. (12.8a) we obtain

$$H[e^{j0}] = \frac{1}{\sqrt{1.64 - 1.6 \cos(0)}} = \frac{1}{\sqrt{0.04}} = 5 = 5\angle 0$$

Therefore

$$|H[e^{j0}]| = 5 \quad \text{and} \quad \angle H[e^{j0}] = 0$$

These values also can be read directly from Figs. 12.1a and 12.1b, respectively, corresponding to $\Omega = 0$. Therefore, the system response to input 1 is

$$y[k] = 5(1^k) = 5 \quad (12.10)$$

(b) $f[k] = \cos\left[\frac{\pi}{6}k - 0.2\right]$

Here $\Omega = \frac{\pi}{6}$. According to Eqs. (12.8)

$$|H[e^{j\pi/6}]| = \frac{1}{\sqrt{1.64 - 1.6 \cos \frac{\pi}{6}}} = 1.983$$

$$\angle H[e^{j\pi/6}] = -\tan^{-1} \left[\frac{0.8 \sin \frac{\pi}{6}}{1 - 0.8 \cos \frac{\pi}{6}} \right] = -0.916 \text{ rad.}$$

These values also can be read directly from Figs. 12.1a and 12.1b, respectively, corresponding to $\Omega = \frac{\pi}{6}$. Therefore

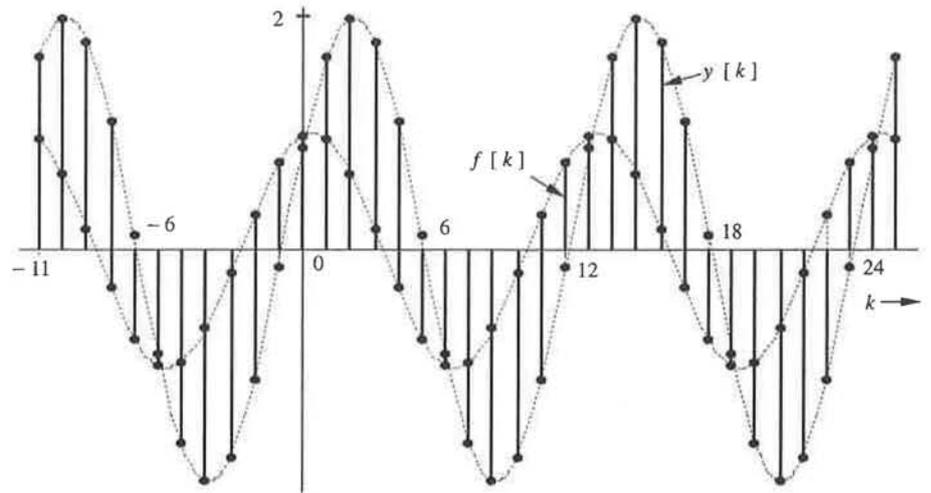


Fig. 12.2 Sinusoidal input and the corresponding output of an LTID system in Example 12.1.

$$\begin{aligned} y[k] &= 1.983 \cos\left(\frac{\pi}{6}k - 0.2 - 0.916\right) \\ &= 1.983 \cos\left(\frac{\pi}{6}k - 1.116\right) \end{aligned} \quad (12.11)$$

Figure 12.2 shows the input $f[k]$ and the corresponding system response.

(c) A sinusoid $\cos 1500t$ sampled every T seconds ($t = kT$) results in a discrete-time sinusoid

$$f[k] = \cos 1500kT \quad (12.12)$$

For $T = 0.001$, the input is

$$f[k] = \cos(1.5k) \quad (12.13)$$

In this case, $\Omega = 1.5$. According to Eqs. (12.8a) and (12.8b)

$$|H[e^{j1.5}]| = \frac{1}{\sqrt{1.64 - 1.6 \cos(1.5)}} = 0.809 \quad (12.14)$$

$$\angle H[e^{j1.5}] = -\tan^{-1} \left[\frac{0.8 \sin(1.5)}{1 - 0.8 \cos(1.5)} \right] = -0.702 \text{ rad} \quad (12.15)$$

These values also could be read directly from Fig. 12.1 corresponding to $\Omega = 1.5$. Therefore

$$y[k] = 0.809 \cos(1.5k - 0.702) \quad \blacksquare$$

⊙ **Computer Example C12.1**

Using MATLAB, find the frequency response of the system in Example 12.1.

```

num=[1 0]; den=[1 -0.8];
W=-pi:pi/100:pi;
H=freqz(num,den,W);
mag=abs(H);
phase=180/pi*unwrap(angle(H));
subplot(2,1,1);
plot(W,mag);grid;
subplot(2,1,2);
plot(W,phase);grid

```

Comment

Figure 12.1 shows plots of amplitude and phase response as functions of Ω . These plots as well as Eqs. (12.8) indicate that the frequency response of a discrete-time system is a continuous (rather than discrete) function of frequency Ω . There is no contradiction here. This behavior is merely an indication of the fact that the frequency variable Ω is continuous (takes on all possible values) and therefore the system response exists at every value of Ω .

△ Exercise E12.1

For a system specified by the equation

$$y[k+1] - 0.5y[k] = f[k]$$

find the amplitude and phase response. Find the system response to sinusoidal input $\cos(1000t - \frac{\pi}{3})$ sampled every $T = 0.5$ ms.

Answer:

$$|H[e^{j\Omega}]| = \frac{1}{\sqrt{1.25 - \cos \Omega}} \quad \angle H[e^{j\Omega}] = -\tan^{-1} \left[\frac{\sin \Omega}{\cos \Omega - 0.5} \right]$$

$$y[k] = 1.639 \cos(0.5k - \frac{\pi}{3} - 0.904) = 1.639 \cos(0.5k - 1.951) \quad \nabla$$

△ Exercise E12.2

Show that for an ideal delay ($H[z] = 1/z$), the amplitude response $|H[e^{j\Omega}]| = 1$, and the phase response $\angle H[e^{j\Omega}] = -\Omega$. Thus, a pure time-delay does not affect the amplitude gain of sinusoidal input, but it causes a phase shift (delay) of Ω radians in a discrete sinusoid of frequency Ω . Thus, in the case of an ideal delay, we see that the phase shift at the output is proportional to the frequency of the input sinusoid (linear phase shift). ∇

The Periodic Nature of the Frequency Response

Figure 12.1 shows that for the system in Example 12.1, the frequency response $H[e^{j\Omega}]$ is a periodic function of Ω with period 2π . This fact is not a coincidence. Unlike the frequency response of a continuous-time system, the frequency response of every LTID system is a periodic function of Ω with period 2π . This fact follows from the very structure of the frequency response $H[e^{j\Omega k}]$. Its argument $e^{j\Omega k}$ is a periodic function of Ω with period 2π . This fact will automatically render $H[e^{j\Omega k}]$ periodic. There is a physical reason for this periodicity and the periodicity of $H[e^{j\Omega k}]$ should not come as a surprise. We know that discrete-time sinusoids separated by values of Ω in integral multiples of 2π are identical. Therefore, the system response to such sinusoids (or exponentials) is also identical. Thus for discrete-time systems, we need to plot the frequency response only over the frequency range from $-\pi$ to π (or from 0 to 2π). In a real sense, discrete-time sinusoids of frequencies outside the fundamental range of frequencies do not exist (although they exist in a technical sense).

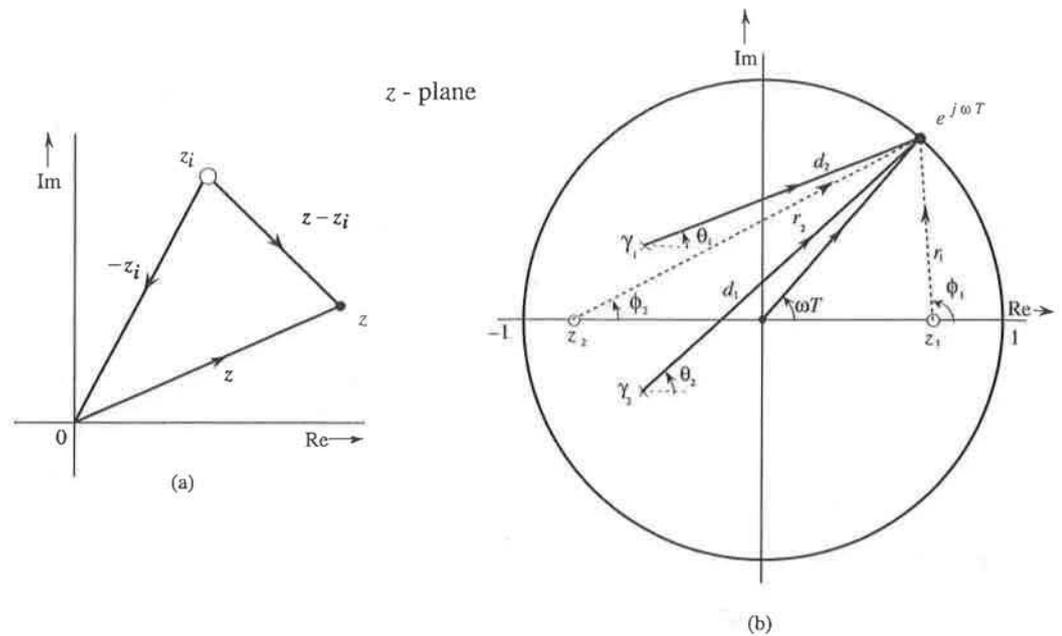


Fig. 12.3 (a) vector representation of complex numbers (b) vector representation of factors of $H[z]$.

12.2 Frequency Response From Pole-Zero Location

The frequency response (amplitude and phase response) of a system are determined by pole-zero locations of the transfer function $H[z]$. Just as in continuous-time systems, it is possible to determine quickly the amplitude and the phase response and to obtain physical insight into the filter characteristics of a discrete-time system by using a graphical technique. Consider the transfer function

$$H[z] = b_n \frac{(z - z_1)(z - z_2) \cdots (z - z_n)}{(z - \gamma_1)(z - \gamma_2) \cdots (z - \gamma_n)} \quad (12.16)$$

We can compute $H[z]$ graphically using the concepts discussed in Sec. 7.3. The directed line segment from z_i to z in the complex plane (Fig. 12.3a) represents the complex number $z - z_i$. The length of this segment is $|z - z_i|$ and its angle with the horizontal axis is $\angle(z - z_i)$.

In filtering applications, the inputs are often the sampled continuous-time sinusoids. Earlier, we showed that a sampled continuous-time sinusoid $\cos \omega t$ appears as a discrete-time sinusoid $\cos \Omega k$ ($\Omega = \omega T$). The appropriate function for computing the frequency response in such a situation, therefore, is $H[e^{j\omega T}]$ ($\Omega = \omega T$). To compute the frequency response $H[e^{j\omega T}]$ we evaluate $H[z]$ at $z = e^{j\omega T}$. But for $z = e^{j\omega T}$, $|z| = 1$ and $\angle z = \omega T$ so that $z = e^{j\omega T}$ represents a point on the unit circle at an angle ωT with the horizontal. We now connect all zeros (z_1, z_2, \dots, z_n) and all poles ($\gamma_1, \gamma_2, \dots, \gamma_n$) to the point $e^{j\omega T}$, as indicated in Fig. 12.3b. Let r_1, r_2, \dots, r_n be the lengths and $\phi_1, \phi_2, \dots, \phi_n$ be the angles, respectively, of the straight

lines connecting z_1, z_2, \dots, z_n to the point $e^{j\omega T}$. Similarly, let d_1, d_2, \dots, d_n be the lengths and $\theta_1, \theta_2, \dots, \theta_n$ be the angles, respectively, of the lines connecting $\gamma_1, \gamma_2, \dots, \gamma_n$ to $e^{j\omega T}$. Then

$$H[e^{j\omega T}] = H[z]|_{z=e^{j\omega T}} = b_n \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2}) \cdots (r_n e^{j\phi_n})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2}) \cdots (d_n e^{j\theta_n})} \quad (12.17)$$

$$= b_n \frac{r_1 r_2 \cdots r_n}{d_1 d_2 \cdots d_n} e^{j[(\phi_1 + \phi_2 + \cdots + \phi_n) - (\theta_1 + \theta_2 + \cdots + \theta_n)]} \quad (12.18)$$

Therefore

$$\begin{aligned} |H[e^{j\omega T}]| &= b_n \frac{r_1 r_2 \cdots r_n}{d_1 d_2 \cdots d_n} \\ &= b_n \frac{\text{product of the distances of zeros to } e^{j\omega T}}{\text{product of distances of poles to } e^{j\omega T}} \end{aligned} \quad (12.19a)$$

and

$$\begin{aligned} \angle H[e^{j\omega T}] &= (\phi_1 + \phi_2 + \cdots + \phi_n) - (\theta_1 + \theta_2 + \cdots + \theta_n) \\ &= \text{sum of zero angles to } e^{j\omega T} - \text{sum of pole angles to } e^{j\omega T} \end{aligned} \quad (12.19b)$$

In this manner, we can compute the frequency response $H[e^{j\omega T}]$ for any value of ω by selecting the point on the unit circle at an angle ωT corresponding to that value of ω . This point is $e^{j\omega T}$. In summary, to compute the frequency response $H[e^{j\omega T}]$, we connect all poles and zeros to this point and determine $|H[e^{j\omega T}]|$ and $\angle H[e^{j\omega T}]$ using the above equations. We repeat this procedure for all values of ωT from 0 to π to obtain the frequency response.

Controlling Gain by Placement of Poles and Zeros

The nature of the influence of pole and zero locations on the frequency response is similar to that observed in continuous-time systems with a minor difference. In place of the imaginary axis of the continuous-time systems, we have a unit circle in the discrete-time case. The nearer the pole (or zero) is to a point $e^{j\omega T}$ (on the unit circle) representing some frequency ω , the more influence that pole (or zero) wields on the amplitude response at that frequency because the length of the vector joining that pole (or zero) to the point $e^{j\omega T}$ is small. The proximity of a pole (or zero) has similar effect on the phase response. From Eq. (12.19a), it is clear that to enhance the amplitude response at a frequency ω we should place a pole as close as possible to the point $e^{j\omega T}$ (on the unit circle) representing that frequency ω . Similarly, to suppress the amplitude response at a frequency ω , we should place a zero as close as possible to the point $e^{j\omega T}$ on the unit circle. Placing repeated poles or zeros will further enhance their influence.

Total suppression of signal transmission at any frequency can be achieved by placing a zero on the unit circle at a point corresponding to that frequency. This is the principle of the notch (bandstop) filter.

Placing a pole or a zero at the origin does not influence the amplitude response because the length of the vector connecting the origin to any point on the unit circle is unity. However, a pole (a zero) at the origin generates angle $-\omega T$ (ωT) in

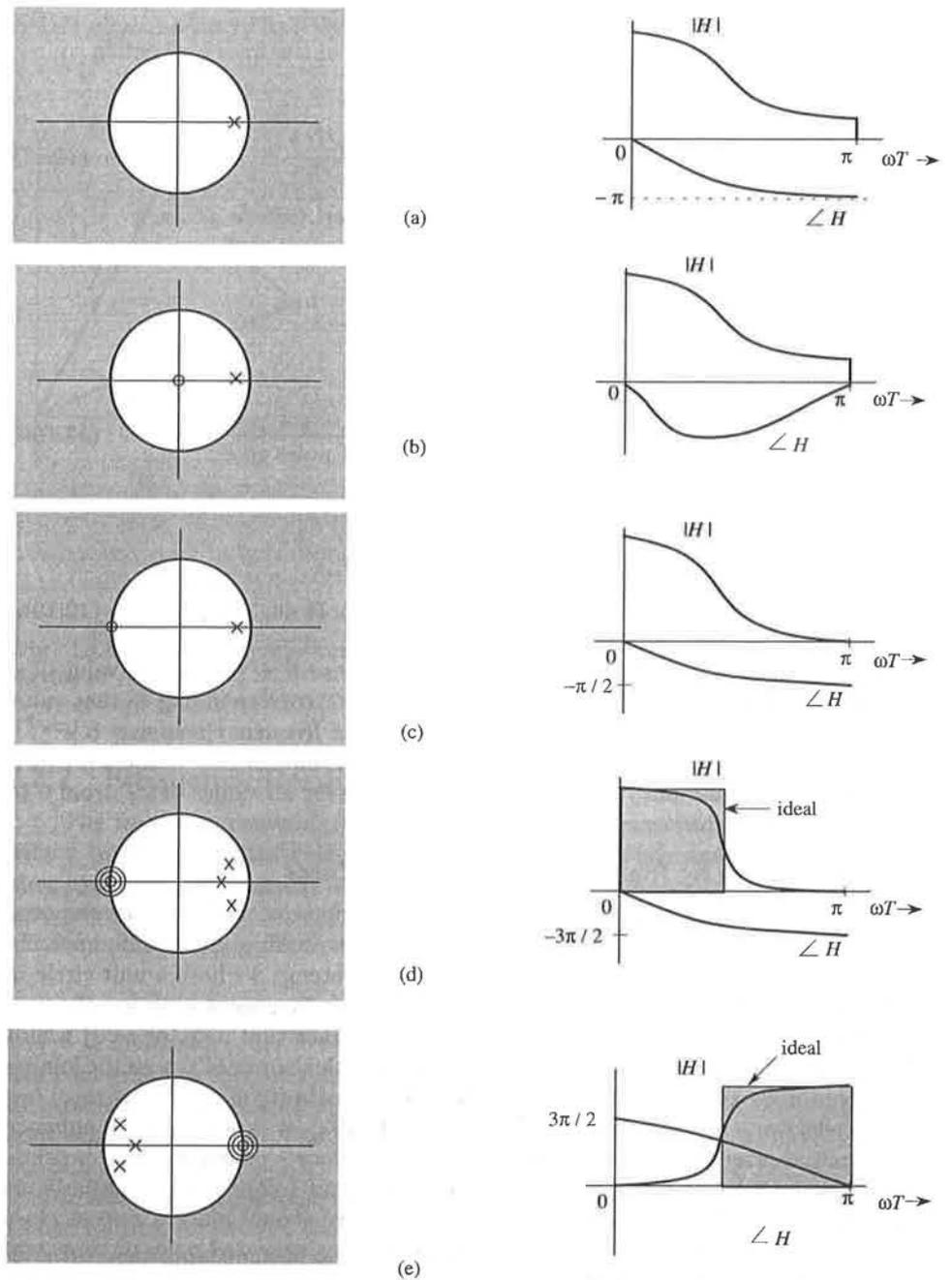


Fig. 12.4 Various pole-zero configurations and the corresponding frequency response.

$\angle H[e^{j\omega T}]$. The phase spectrum $-\omega T$ is a linear function of frequency and therefore represents a pure time-delay of T seconds (see Eq. (10.48) or Exercise E12.2). Therefore, a pole (a zero) at the origin causes a time delay (time advance) of T seconds in the response. There is no change in the amplitude response.

For a stable system, all the poles must be located inside the unit circle. The zeros may lie anywhere. Also, for a physically realizable system, $H[z]$ must be a proper fraction, that is, $n \geq m$. If, to achieve a certain amplitude response, we require $m > n$, we can still make the system realizable by placing a sufficient number of poles at the origin. This will not change the amplitude response but it will increase the time delay of the response.

In general, a pole at a point has the opposite effect of a zero at that point. Placing a zero closer to a pole tends to cancel the effect of that pole on the frequency response.

Lowpass Filters

A lowpass filter has a maximum gain at $\omega = 0$, which corresponds to point $e^{j0T} = 1$ on the unit circle. Clearly, placing a pole inside the unit circle near the point $z = 1$ (Fig. 12.4a) would result in a lowpass response. The corresponding amplitude and phase response appears in Fig. 12.4a. For smaller values of ω , the point $e^{j\omega T}$ (a point on the unit circle at an angle ωT) is closer to the pole, and consequently the gain is higher. As ω increases, the distance of the point $e^{j\omega T}$ from the pole increases. Consequently the gain decreases, resulting in a lowpass characteristic. Placing a zero at the origin does not change the amplitude response but it does modify the phase response, as illustrated in Fig. 12.4b. Placing a zero at $z = -1$, however, changes both the amplitude and phase response (Fig. 12.4c). The point $z = -1$ corresponds to frequency $\omega = \pi/T$ ($z = e^{j\omega T} = e^{j\pi} = -1$). Consequently, the amplitude response now becomes more attenuated at higher frequencies, with a zero gain at $\omega T = \pi$. We can approach ideal lowpass characteristics by using more poles staggered near $z = 1$ (but within the unit circle). Figure 12.4d shows a third-order lowpass filter with three poles near $z = 1$ and a third-order zero at $z = -1$, with corresponding amplitude and phase response. For an ideal lowpass filter we need an enhanced gain at every frequency in the band $(0, \omega_c)$. This can be achieved by placing a continuous wall of poles (requiring an infinite number of poles) opposite this band.

Highpass Filters

A highpass filter has a small gain at lower frequencies and a high gain at higher frequencies. Such a characteristic can be realized by placing a pole or poles near $z = -1$ because we want the gain at $\omega T = \pi$ to be the highest. Placing a zero at $z = 1$ further enhances suppression of gain at lower frequencies. Figure 12.4e shows a possible pole-zero configuration of the third-order highpass filter with corresponding amplitude and phase response.

■ Example 12.2: Bandpass Filter

Using trial-and-error, design a tuned (bandpass) filter with zero transmission at 0 Hz and also at 500 Hz. The resonant frequency is required to be 125 Hz. The highest frequency to be processed is $\mathcal{F}_h = 500$ Hz.

Because $\mathcal{F}_h = 500$, we require $T \leq \frac{1}{1000}$ [see Eq. (8.17)]. Let us select $T = 10^{-3}$. Since the amplitude response is zero at $\omega = 0$ and $\omega = 1000\pi$, we need to place zeros at $e^{j\omega T}$ corresponding to $\omega = 0$ and $\omega = 1000\pi$. For $\omega = 0$, $z = e^{j\omega T} = 1$; for $\omega = 1000\pi$ (with $T = 10^{-3}$), $e^{j\omega T} = -1$. Hence, there must be zeros at $z = \pm 1$. Moreover, we need enhanced frequency response at $\omega = 250\pi$. This frequency (with $\omega T = \pi/4$) corresponds to $z = e^{j\omega T} = e^{j\pi/4}$. Therefore, to enhance the frequency response at this frequency, we

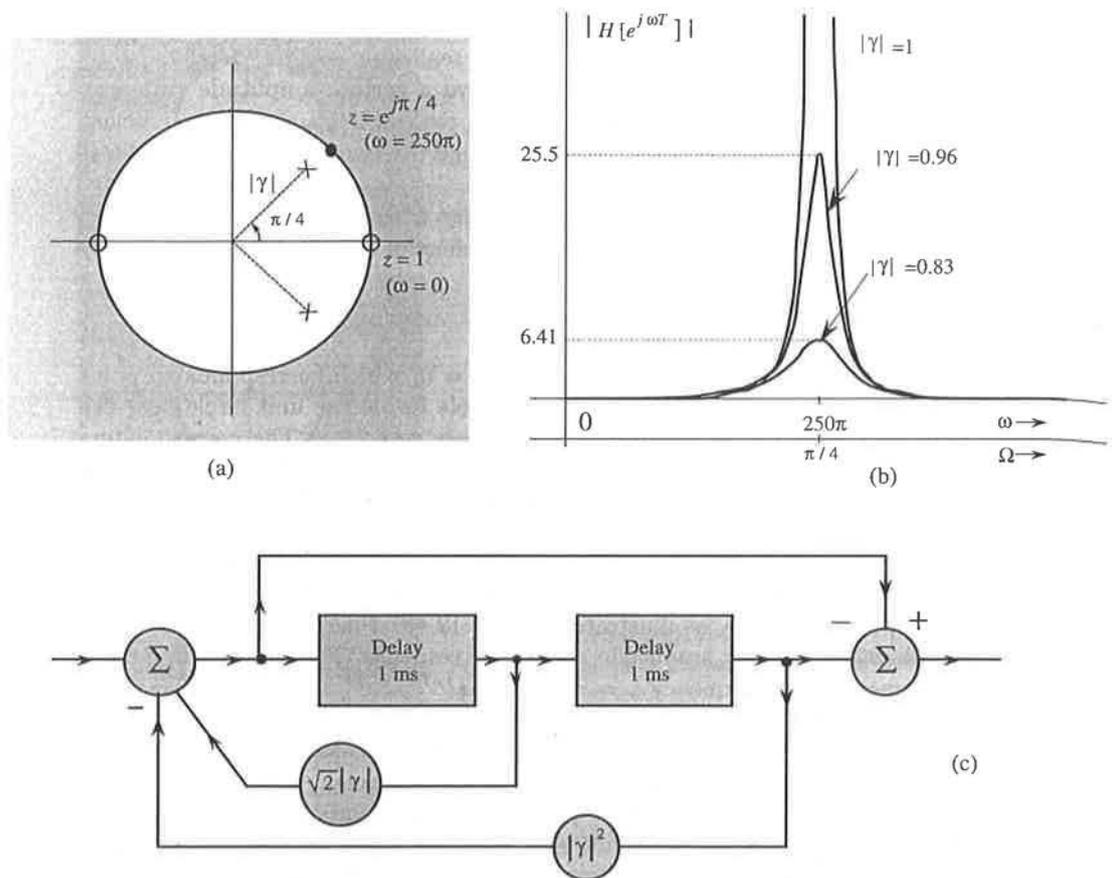


Fig. 12.5 Designing a bandpass filter in Example 12.2.

place a pole in its vicinity (near $e^{j\pi/4}$). Because this is a complex pole we also need its conjugate near $e^{-j\pi/4}$, as indicated in Fig. 12.5a. Let us choose these poles λ_1 and λ_2 as

$$\lambda_1 = |\gamma|e^{j\pi/4} \quad \text{and} \quad \lambda_2 = |\gamma|e^{-j\pi/4}$$

where $|\gamma| < 1$ for stability. The closer the value of $|\gamma|$ is to the unit circle, the more sharply peaked is the response around $\omega = 250\pi$. We also have a zeros at ± 1 . Hence

$$H[z] = K \frac{(z-1)(z+1)}{(z-|\gamma|e^{j\pi/4})(z-|\gamma|e^{-j\pi/4})} = K \frac{z^2-1}{z^2-\sqrt{2}|\gamma|z+|\gamma|^2} \quad (12.20)$$

For convenience we shall choose $K = 1$. The amplitude response is given by

$$|H[e^{j\omega T}]| = \frac{|e^{j2\omega T} - 1|}{|e^{j\omega T} - |\gamma|e^{j\pi/4}| |e^{j\omega T} - |\gamma|e^{-j\pi/4}|}$$

Now, using Eq.(12.9), we obtain

$$|H[e^{j\omega T}]|^2 = \frac{2(1 - \cos 2\omega T)}{[1 + |\gamma|^2 - 2|\gamma| \cos(\omega T - \frac{\pi}{4})][1 + |\gamma|^2 - 2|\gamma| \cos(\omega T + \frac{\pi}{4})]} \quad (12.21)$$

Figure 12.5b shows the amplitude response for values of $|\gamma| = 0.83, 0.96,$ and 1 . As expected, the gain is zero at $\omega = 0$ and at 500 Hz ($\omega = 1000\pi$). The gain peaks at about 125 Hz ($\omega = 250\pi$). The resonance (peaking) becomes pronounced as $|\gamma|$ approaches 1 . Fig. 12.5c shows a canonical realization of this filter [see Eq. (12.20)]. ■

⊙ **Computer Example C12.2**

Using MATLAB, find the frequency response of the bandpass filter in Example 12.2 for $|\gamma| = 0.96$.

```
gamma=0.96;
num=[1 0 -1]; den=[1 -sqrt(2)*gamma gamma^2];
W=-pi:pi/100:pi;
H=freqz(num,den,W);
mag=abs(H);
phase=180/pi*unwrap(angle(H));
subplot(2,1,1)
plot(W,mag)
subplot(2,1,2)
plot(W,phase) ⊙
```

■ **Example 12.3: Notch (Bandstop) Filter**

Design a second-order notch filter to have zero transmission at 250 Hz and a sharp recovery of gain to unity on both sides of 250 Hz. The highest significant frequency to be processed is $\mathcal{F}_h = 500$ Hz.

In this case, $T \leq 1/2\mathcal{F}_h = 10^{-3}$. Let us choose $T = 10^{-3}$. For the frequency 250 Hz, $\omega T = 2\pi(250)T = \pi/2$. Thus, the frequency 250 Hz is represented by a point $e^{j\omega T} = e^{j\pi/2} = j$ on the unit circle, as depicted in Fig. 12.6a. Since we need zero transmission at this frequency, we must place a zero at $z = e^{j\pi/2} = j$ and its conjugate at $z = e^{-j\pi/2} = -j$. We also require a sharp recovery of gain on both sides of frequency 250 Hz. To accomplish this goal, we place two poles close to the two zeros in order to cancel out the effect of the zeros as we move away from the point j (corresponding to frequency 250 Hz). For this reason, let us use poles at $\pm ja$ with $a < 1$ for stability. The closer the poles are to the zeros (the closer the a to 1), the faster is the gain recovery on either side of 250 Hz. The resulting transfer function is

$$H[z] = K \frac{(z-j)(z+j)}{(z-ja)(z+ja)} = K \frac{z^2 + 1}{z^2 + a^2}$$

The dc gain (gain at $\omega = 0$, or $z = 1$) of this filter is

$$H[1] = K \frac{2}{1+a^2}$$

Because we require a dc gain of unity, we must select $K = \frac{1+a^2}{2}$. The transfer function is therefore

$$H[z] = \frac{(1+a^2)(z^2+1)}{2(z^2+a^2)} \quad (12.22)$$

and according to Eq. (12.9)

$$|H[e^{j\omega T}]|^2 = \frac{(1+a^2)^2}{4} \frac{(e^{j2\omega T} + 1)(e^{-j2\omega T} + 1)}{(e^{j2\omega T} + a^2)(e^{-j2\omega T} + a^2)} = \frac{(1+a^2)^2(1 + \cos 2\omega T)}{2(1+a^4 + 2a^2 \cos 2\omega T)}$$

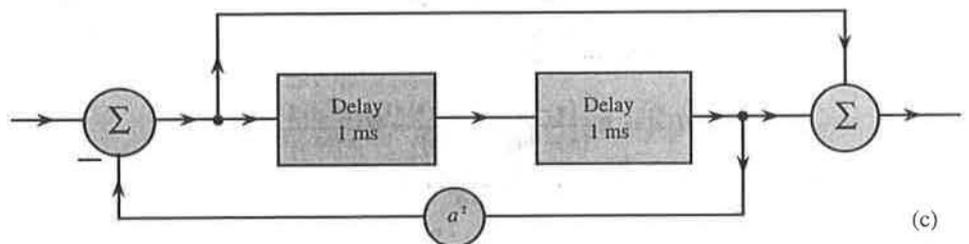
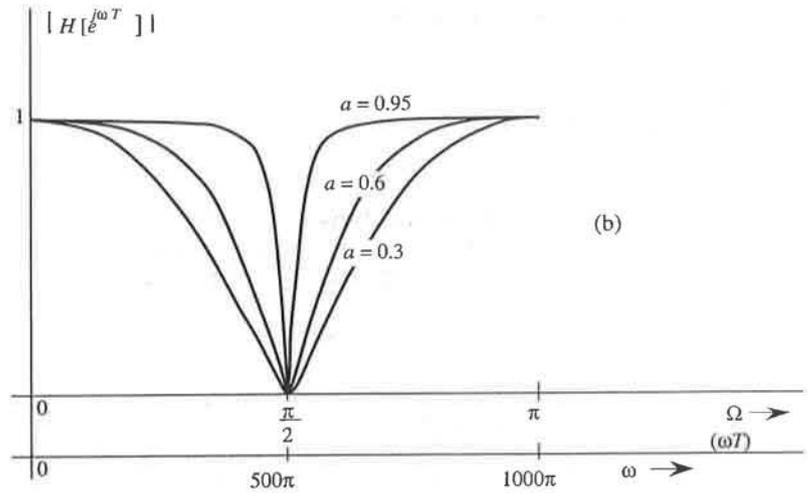
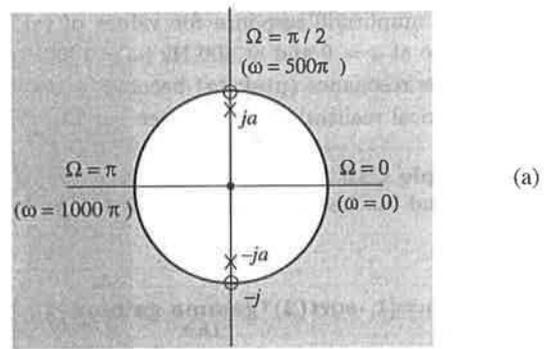


Fig. 12.6 Designing a notch (bandstop) filter in Example 12.3.

Figure 12.6b shows $|H[e^{j\omega T}]|$ for values of $a = 0.3, 0.6,$ and 0.95 . Figure 12.6c shows a realization of this filter. ■

△ **Exercise E12.3**

Using the graphical argument, show that a filter with transfer function

$$H[z] = \frac{z - 0.9}{z}$$

acts as a highpass filter. Make a rough sketch of the amplitude response. ▽

12.3 Digital Filters

Digital filters and some of their advantages were discussed in Sec. 8.5. Digital filters can be classified as either **recursive** or **nonrecursive**.

Recursive Filters

The terms *recursive* and *nonrecursive* are easily understood in terms of a specific example. Consider a third-order system with the transfer function

$$H[z] = \frac{b_3z^3 + b_2z^2 + b_1z + b_0}{z^3 + a_2z^2 + a_1z + a_0} \quad (12.23)$$

The input $f[k]$ and the corresponding output $y[k]$ of this system are related by the difference equation.

$$y[k] + a_2y[k-1] + a_1y[k-2] + a_0y[k-3] = b_3f[k] + b_2f[k-1] + b_1f[k-2] + b_0f[k-3] \quad (12.24a)$$

or

$$y[k] = \underbrace{-a_2y[k-1] - a_1y[k-2] - a_0y[k-3]}_{\text{output terms}} + \underbrace{b_3f[k] + b_2f[k-1] + b_1f[k-2] + b_0f[k-3]}_{\text{input terms}} \quad (12.24b)$$

Here $y[k]$, the output at instant k , is determined by the input values $f[k]$, $f[k-1]$, $f[k-2]$, and $f[k-3]$ as well as by the past output values $y[k-1]$, $y[k-2]$, and $y[k-3]$. The output is therefore determined iteratively or recursively from the its past values. To compute the present output of this third-order system, we use the past three values of the output. In general, for an n th-order system we use the past n values of the output. Such a system is called a **recursive system**.

An interesting feature of a recursive system is that once an output exists, it tends to propagate itself forever because of its recursive nature. This is also seen from the canonical realization of $H[z]$ in Fig. 12.7a. Once an input (any input) is applied, the feedback connections loop the output continuously back into the system, and the output continues forever. This propagation of the output in perpetuity occurs because of the nonzero values of coefficients $a_0, a_1, a_2, \dots, a_{n-1}$. These coefficients [appearing in the denominator of $H[z]$ in Eq. (12.23)] are the **recursive coefficients**. The coefficients $b_0, b_1, b_2, \dots, b_n$ (appearing in the numerator of $H[z]$) are the **nonrecursive coefficients**. If an input $\delta[k]$ is applied at the input of a recursive filter, the response $h[k]$ will continue forever up to $k = \infty$. For this reason, recursive filters are also known as **infinite impulse response (IIR)** filters.

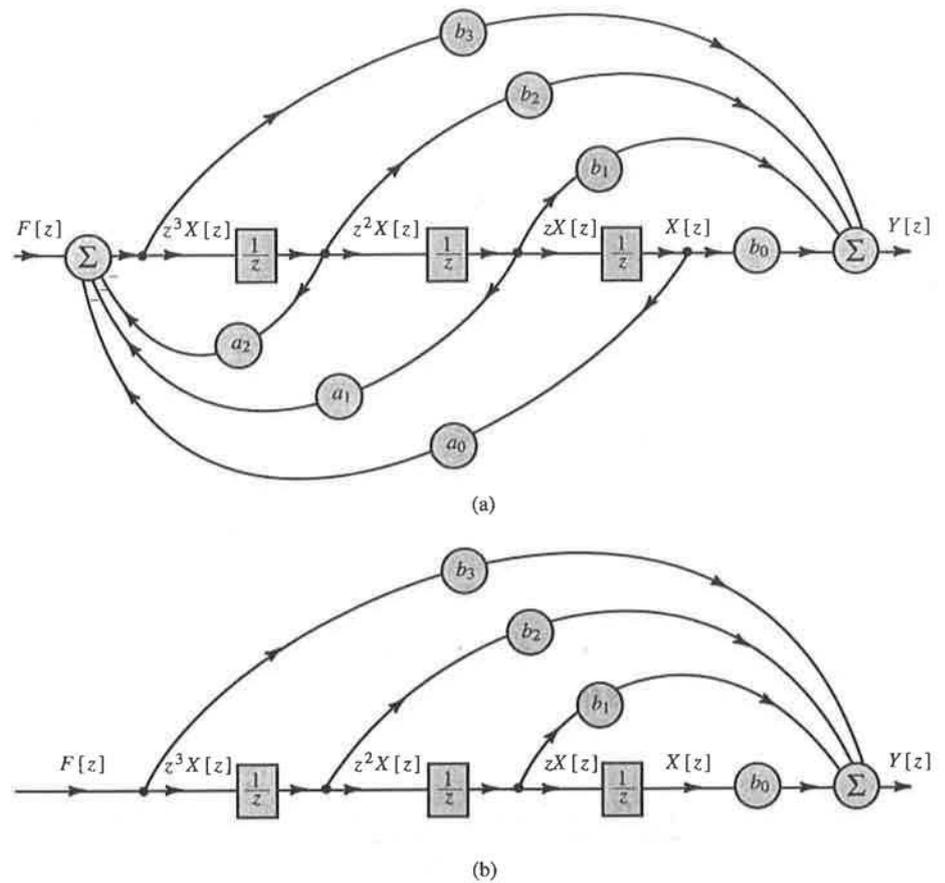


Fig. 12.7 Digital filter realization: (a) recursive filter (b) nonrecursive filter.

Nonrecursive Filters

If the recursive coefficients a_0 , a_1 , and a_2 are zero, $H[z]$ in Eq. (12.23) reduces to

$$H[z] = \frac{b_3z^3 + b_2z^2 + b_1z + b_0}{z^3} \quad (12.25a)$$

$$= b_3 + \frac{b_2}{z} + \frac{b_1}{z^2} + \frac{b_0}{z^3} \quad (12.25b)$$

The difference equation corresponding to this system now reduces to

$$y[k] = b_3f[k] + b_2f[k-1] + b_1f[k-2] + b_0f[k-3] \quad (12.26)$$

Note that $y[k]$ is now computed from the present and the three past values of the input $f[k]$. Such filters are called **nonrecursive filters**. Figure 12.7b shows a canonical realization of $H[z]$, which is identical to the realization in Fig. 12.7a, with all the feedback connections eliminated. If we apply an input $f[k] = \delta[k]$ to this

system, the input will pass through the system and will be completely out of the system by $k = 4$. There are no feedback connections to perpetuate the output. Therefore, $h[k]$ will be nonzero only for $k = 0, 1, 2,$ and 3 . For an n th-order nonrecursive filter, $h[k]$ is zero for $k > n$. Therefore, the duration of $h[k]$ is finite for a nonrecursive filter. For this reason, nonrecursive filters are also known as **finite impulse response (FIR)** filters. Nonrecursive filters are a special case of recursive filters, in which all the recursive coefficients $a_0, a_1, a_2, \dots, a_{n-1}$ are zero.

An n th-order nonrecursive filter transfer function is given by ($a_0 = a_1 = a_2 = \dots = a_{n-1} = 0$).

$$H[z] = \frac{b_n z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0}{z^n} \quad (12.27a)$$

$$= b_n + \frac{b_{n-1}}{z} + \frac{b_{n-2}}{z^2} + \dots + \frac{b_1}{z^{n-1}} + \frac{b_0}{z^n} \quad (12.27b)$$

The inverse z -transform of this equation yields

$$h[k] = b_n \delta[k] + b_{n-1} \delta[k-1] + \dots + b_1 \delta[k-n+1] + b_0 \delta[k-n] \quad (12.28)$$

Observe that $h[k] = 0$ for $k > n$.

Because nonrecursive filters are a special case of recursive filters, we expect the performance of recursive filters to be superior. This expectation is true in the sense that a given amplitude response can be achieved by a recursive filter of an order smaller than that required for the corresponding nonrecursive filter. However, nonrecursive filters have the advantage of having linear phase characteristics. Recursive filters can realize linear phase only approximately.

12.4 Filter Design Criteria

A digital filter processes discrete-time signals to yield a discrete-time output. Digital filters can also process analog signals by converting them into discrete-time signals. If the input is a continuous-time signal $f(t)$, it is converted into a discrete-time signal $f[k] = f(kT)$ by a C/D (continuous-time to discrete-time) converter. The signal $f[k]$ is now processed by a "digital" (meaning discrete-time) system with transfer function $H[z]$. The output $y[k]$ of $H[z]$ is then converted into an "analog" (meaning continuous-time) signal $y(t)$. The system in Fig. 12.8a, therefore, acts as a continuous-time (or "analog") system. Our objective is to determine the "digital" (discrete-time) processor $H[z]$ that will make the system in Fig. 12.8a equivalent to a desired "analog" (continuous-time) system with transfer function $H_a(s)$, shown in Fig. 12.8b.

We may strive to make the two systems behave similarly in the time-domain or in the frequency-domain. Accordingly, we have two different design procedures. Let us now determine the equivalence criterion of the two systems in the time-domain and in the frequency-domain.

12.4-1 The Time-Domain Equivalence Criterion

By time-domain equivalence we mean that for the same input $f(t)$, the output $y(t)$ of the system in Fig. 12.8a is equal to the output $y(t)$ of the system in Fig.

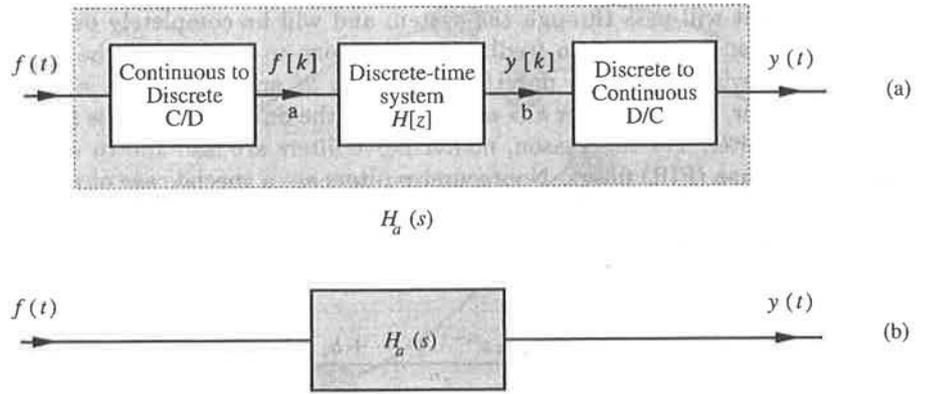


Fig. 12.8 Analog filter realization with a digital filter.

12.8b. Therefore $y(kT)$, the samples of the output in Fig. 12.8b, are identical to $y[k]$, the output of $H[z]$ in Fig. 12.8a.

The output $y(t)$ of the system in Fig. 12.8b is†

$$y(t) = \int_{-\infty}^{\infty} f(\tau)h_a(t - \tau) d\tau$$

An integral is a sum in the limit. Therefore, the above equation can be expressed as

$$y(t) = \lim_{\Delta\tau \rightarrow 0} \sum_{m=-\infty}^{\infty} f(m\Delta\tau)h_a(t - m\Delta\tau)\Delta\tau \quad (12.29a)$$

For our purpose it is convenient to use the notation T for $\Delta\tau$ in the above equation. Such a change of notation yields

$$y(t) = \lim_{T \rightarrow 0} T \sum_{m=-\infty}^{\infty} f(mT)h_a(t - mT) \quad (12.29b)$$

The response at the k th sampling instant is $y(kT)$ obtained by setting $t = kT$ in the above equation:

$$y(kT) = \lim_{T \rightarrow 0} T \sum_{m=-\infty}^{\infty} f(mT)h_a[(k - m)T] \quad (12.29c)$$

In Fig. 12.8a, the input to $H[z]$ is $f(kT) = f[k]$. If $h[k]$ is the unit impulse response of $H[z]$, then $y[k]$, the output of $H[z]$, is given by

$$y[k] = \sum_{m=-\infty}^{\infty} f[m]h[k - m] \quad (12.30)$$

†For the sake of generality, we are assuming a noncausal system. The argument and the results are also valid for causal systems.

If the two systems are to be equivalent, $y(kT)$ in Eq. (12.29c) must be equal to $y[k]$ in Eq. (12.30). Therefore

$$h[k] = \lim_{T \rightarrow 0} T h_a(kT) \quad (12.31)$$

This is the time-domain criterion for equivalence of the two systems. according to this criterion, $h[k]$, the unit impulse response of $H[z]$ in Fig. 12.8a, must be equal to T times the samples of $h_a(t)$, the unit impulse response of the system in Fig. 12.8b, assuming that $T \rightarrow 0$. This is known as the **impulse invariance criterion** of filter design.†

12.4-2 The Frequency-Domain Equivalence Criterion

In Sec. 2.4-3 [Eq. (2.47)], we proved that for an analog system with transfer function $H_a(s)$, the system response $y(t)$ to the everlasting exponential input $f(t) = e^{st}$ is also an everlasting exponential

$$y(t) = H_a(s)e^{st} \quad (12.32)$$

Similarly, in Eq. (9.57a), we showed that for a discrete-time system with transfer function $H[z]$, the system response $y[k]$ to an everlasting exponential input $f[k] = z^k$ is also an everlasting exponential $H[z]z^k$:

$$y[k] = H[z]z^k \quad (12.33)$$

If the systems in Figs. 12.8a and 12.8b are equivalent, then the response of both systems to an everlasting exponential input $f(t) = e^{st}$ should be the same. A continuous-time signal $f(t) = e^{st}$ sampled every T seconds results in a discrete-time signal

$$\begin{aligned} f[k] &= e^{skT} \\ &= z^k \quad \text{with } z = e^{sT} \end{aligned}$$

This discrete-time exponential z^k is applied at the input of $H[z]$ in Fig. 12.8a, whose response is

$$\begin{aligned} y[k] &= H[z]z^k \Big|_{z=e^{sT}} \\ &= H[e^{sT}]e^{skT} \end{aligned} \quad (12.34)$$

Now, for the system in Fig. 12.8b, $y(kT)$, the k th sample of the output $y(t)$ in Eq. (12.32), is

$$y(kT) = H_a(s)e^{skT} \quad (12.35)$$

If the two systems are to be equivalent, a necessary condition is that $y[k]$ in Eq. (12.34) must be equal to $y(kT)$ in Eq. (12.35). This condition means that

$$H[e^{sT}] = H_a(s) \quad (12.36)$$

This is the frequency-domain criterion for equivalence of the two systems. It should be remembered, however, that with this criterion we are ensuring only that the

†Because T is a constant, some authors ignore the factor T , which yields alternate criterion $h[k] = h_a(kT)$. Ignoring T merely scales the amplitude response of the resulting filter.

digital filter's response matches exactly that of the desired analog filter at sampling instants. If we want the two responses to match at every value of t , we must have $T \rightarrow 0$. Therefore

$$\lim_{T \rightarrow 0} H[e^{sT}] = H_a(s) \quad (12.37)$$

A Practical Difficulty

Both of these criteria for filter design require the condition $T \rightarrow 0$ for realizing a digital filter equivalent to a given analog filter. However, this condition is impossible in practice because it necessitates an infinite sampling rate, resulting in an infinite data rate. In practice, we must choose a small but nonzero T to achieve a compromise between the two conflicting requirements, namely closeness of approximation and system cost.

This approximation, however, does not mean that the system in Fig. 12.8a is inferior to that in Fig. 12.8b, because often $H_a(s)$ itself is an approximation to what we are seeking. For example, in lowpass filter design we strive to design a system with ideal lowpass characteristics. Failing that, however, we settle for some approximation such as Butterworth lowpass transfer functions. In fact, it is entirely possible that $H[z]$, which is an approximation to $H_a(s)$, may be a better approximation to the desired characteristics than is $H_a(s)$ itself.

12.5 Recursive Filter Design by the Time-Domain Criterion: The Impulse Invariance Method

The time-domain design criterion for the equivalence of the systems in Figs. 12.8a and 12.8b is [see Eq. (12.31)]

$$h[k] = \lim_{T \rightarrow 0} T h_a(kT) \quad (12.38)$$

where $h[k]$ is the unit impulse response of $H[z]$, $h_a(t)$ is the unit impulse response of $H_a(s)$, and T is the sampling interval in Fig. 12.8a.

As indicated earlier, it is impractical to let $T \rightarrow 0$. In practice, T is chosen to be small but nonzero. We have already discussed the effect of aliasing and the consequent distortion in the frequency response caused by nonzero T . Assuming that we have selected a suitable value of T , we can ignore the condition $T \rightarrow 0$, and Eq. (12.38) can be expressed as

$$h[k] = T h_a(kT) \quad (12.39)$$

The z -transform of this equation yields

$$H[z] = TZ \left(h_a(kT) \right) \quad (12.40)$$

This result yields the desired transfer function $H[z]$.

Let us consider a first-order transfer function

$$H_a(s) = \frac{c}{s - \lambda} \quad (12.41a)$$

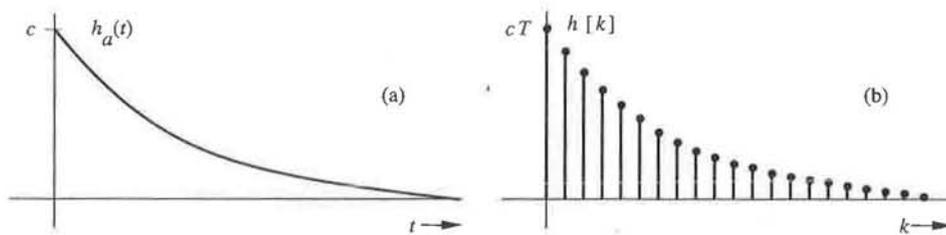


Fig. 12.9 Procedure for the impulse invariance method of filter design.

The impulse response $h(t)$ of this filter is the inverse Laplace transform of $H_a(s)$, which in this case is

$$h_a(t) = ce^{\lambda t}u(t) \quad (12.41b)$$

The corresponding digital filter unit sample response $h[k]$ is given by Eq. (12.39)

$$h[k] = Th_a(kT) = Tce^{k\lambda T} \quad (12.42)$$

Figures 12.9a and b show $h_a(t)$ and $h[k]$. According to Eq. (12.40), $H[z]$ is T times the z -transform of $h[k]$. Thus,

$$H[z] = \frac{Tcz}{z - e^{\lambda T}} \quad (12.43)$$

The procedure of finding $H[z]$ can be systematized for any n th-order system. First we express an n th-order analog transfer function $H_a(s)$ as a sum of partial fractions as

$$H_a(s) = \sum_{i=1}^n \frac{c_i}{s - \lambda_i} \quad (12.44)$$

Then the corresponding $H[z]$ is given by

$$H[z] = T \sum_{i=1}^n \frac{c_i z}{z - e^{\lambda_i T}} \quad (12.45)$$

This transfer function can be readily realized as a parallel combination of the n first-order systems if all the n poles of $H_a(s)$ are real. The complex conjugate poles, if any, must be realized as a single second-order term. Table 12.1 lists several pairs of $H_a(s)$ and their corresponding $H[z]$.

Choosing the Sampling Interval T

If \mathcal{F}_h is the highest frequency to be processed, then the sampling interval must be no greater than $1/2\mathcal{F}_h$ in order to avoid signal aliasing. However, in the impulse invariance method, there is yet another consideration, which must also be taken into account. Consider a hypothetical frequency response $H_a(j\omega)$ (Fig. 12.10a) that we wish to realize using a digital filter, as illustrated in Fig. 12.8a. Let us assume that we have an equivalent digital filter that meets the time-domain equivalence criterion in Eq. (12.37); that is,

TABLE 12.1

	$H_a(s)$	$h_a(t)$	$h[k]$	$H[z]$
1	K	$K\delta(t)$	$TK\delta[k]$	TK
2	$\frac{1}{s}$	$u(t)$	$Tu[k]$	$\frac{Tz}{z-1}$
3	$\frac{1}{s^2}$	t	kT^2	$\frac{T^2z}{(z-1)^2}$
4	$\frac{1}{s^3}$	$\frac{t^2}{2}$	$\frac{k^2T^3}{2}$	$\frac{T^3z(z+1)}{2(z-1)^3}$
5	$\frac{1}{s-\lambda}$	$e^{\lambda t}$	$Te^{\lambda kT}$	$\frac{Tz}{z-e^{\lambda T}}$
6	$\frac{1}{(s-\lambda)^2}$	$te^{\lambda t}$	$kT^2e^{\lambda kT}$	$\frac{T^2ze^{\lambda T}}{(z-e^{\lambda T})^2}$
7	$\frac{As+B}{s^2+2as+c}$	$Tr e^{-at} \cos(bt+\theta)$	$Tr e^{-akT} \cos(bkT+\theta)$	$\frac{Trz [z \cos \theta - e^{-aT} \cos(bT-\theta)]}{z^2 - (2e^{-aT} \cos bT)z + e^{-2aT}}$
	$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$	$b = \sqrt{c-a^2}$	$\theta = \tan^{-1} \left(\frac{Aa-B}{A\sqrt{c-a^2}} \right)$	

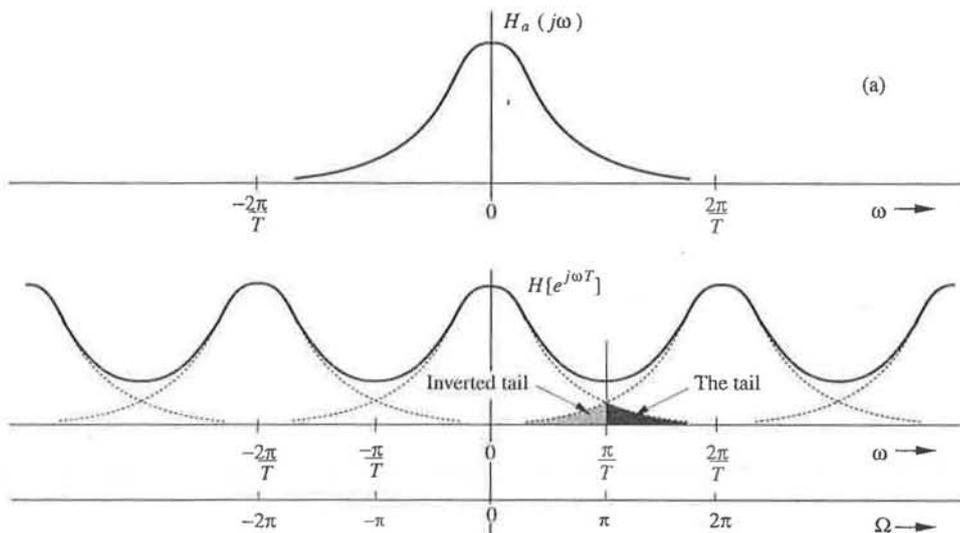


Fig. 12.10 Aliasing in digital filters, and a choice of the sampling interval T .

$$h[k] = \lim_{T \rightarrow 0} T h_a(kT)$$

In Chapter 5 (Fig. 5.6), we showed that the Fourier transform of the samples of $h_a(t)$ consists of periodic repetition of $H_a(j\omega)$ with period equal to the sampling frequency $\omega_s = 2\pi/T = 2\pi\mathcal{F}_s$.[†] Also $H_a(j\omega)$ is not generally bandlimited. Hence, aliasing among various repeating cycles cannot be prevented, as depicted in Fig. 12.10b. The resulting spectrum will be different from the desired spectrum, especially at higher frequencies. If $H_a(j\omega)$ were to be bandlimited; that is, if $H_a(j\omega) = 0$ for $|\omega| > \omega_0$, then the overlap could be avoided if we select the period $2\pi/T > 2\omega_0$. However, according to the Paley-Wiener criterion [Eq. (4.61)], every practical system frequency response is nonbandlimited, and the cycle overlap is inevitable. However, for frequencies beyond some ω_0 , if $|H_a(j\omega)|$ is a negligible fraction, say 1%, of $H_a(j\omega)|_{\max}$, then we can consider¹ $H_a(j\omega)$ to be essentially bandlimited to ω_0 , and we can select

$$T = \frac{\pi}{\omega_0} \quad (12.46)$$

Example 12.4

Design a digital filter to realize the first-order analog lowpass Butterworth filter with the transfer function

$$H_a(s) = \frac{\omega_c}{s + \omega_c} \quad \omega_c = 10^5 \quad (12.47)$$

[†]How can we apply the discussion in Chapter 5, which applies to impulse samples of continuous-time signals, to discrete-time signals? Recall our discussion in Sec. 10.4 (Fig. 10.8), where we showed that the spectrum of discrete-time signal is just a scaled version of the spectrum of the impulse samples of the corresponding continuous-time signal.

For this filter, we find the corresponding $H[z]$ according to Eq. (12.43) (or pair 5 in Table 12.1) as

$$H[z] = \frac{\omega_c T z}{z - e^{-\omega_c T}} \quad (12.48)$$

Next, we select the value of T according to Eq. (12.46). We find the essential filter bandwidth ω_0 , where the filter gain is 1% of the maximum filter gain. Here we use

$$|H_a(j\omega)| = \left| \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}} \right|$$

In this case $|H_a(j\omega)|_{\max} = 1$. Hence, the essential bandwidth ω_0 is that frequency where the $|H_a(j\omega_0)| = 0.01$. Observe that

$$|H_a(j\omega)| \approx \frac{\omega_c}{\omega} \quad \omega \gg \omega_c$$

Hence,

$$|H_a(j\omega_0)| \approx \frac{\omega_c}{\omega_0} = 0.01 \implies \omega_0 = 100\omega_c = 10^7$$

Thus, for good results, we should select $T = \frac{\pi}{\omega_0} = 10^{-7}\pi$. However, for the sake of demonstrating aliasing effect of the overlapping cycles, we shall deliberately select a lower value of ω_0 (higher T). Let us select

$$T = \left(\frac{\pi}{10\omega_c} \right) = 10^{-6}\pi \quad \text{and} \quad \omega_c T = \frac{\pi}{10} \quad (12.49)$$

Substitution of this value in Eq. (12.48) yields

$$H[z] = \frac{0.3142z}{z - 0.7304} \quad (12.50)$$

A canonical realization of this filter is shown in Fig. 12.11a by following the procedure in Sec. 11.4 (see Example 6.18c, Fig. 6.25b). Note the recursive nature of the filter. To find the frequency response of this digital filter, we rewrite $H[z]$ as

$$H[z] = \frac{0.3142}{1 - 0.7304z^{-1}}$$

Therefore

$$H[e^{j\omega T}] = \frac{0.3142}{1 - 0.7304e^{-j\omega T}} = \frac{0.3142}{(1 - 0.7304 \cos \omega T) + j0.7304 \sin \omega T}$$

Consequently

$$|H[e^{j\omega T}]| = \frac{0.3142}{\sqrt{(1 - 0.7304 \cos \omega T)^2 + (0.7304 \sin \omega T)^2}} = \frac{0.3142}{\sqrt{1.533 - 1.4608 \cos \omega T}}$$

$$\angle H[e^{j\omega T}] = -\tan^{-1} \left(\frac{0.7304 \sin \omega T}{1 - 0.7304 \cos \omega T} \right)$$

Substituting $T = \pi \times 10^{-6}$ in the above equations, we obtain

$$|H[e^{j\omega T}]| = \frac{0.1\pi}{\sqrt{1.533 - 1.4608 \cos(\pi \times 10^{-6}\omega)}} \quad (12.51a)$$

$$\angle H[e^{j\omega T}] = -\tan^{-1} \left[\frac{0.7304 \sin(\pi \times 10^{-6}\omega)}{1 - 0.7304 \cos(\pi \times 10^{-6}\omega)} \right] \quad (12.51b)$$

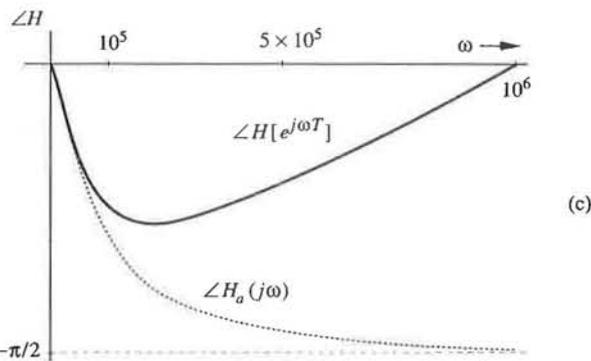
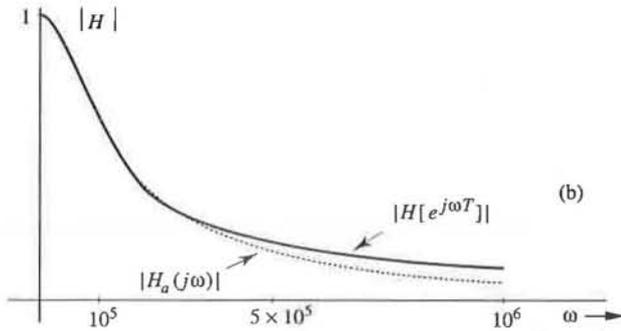
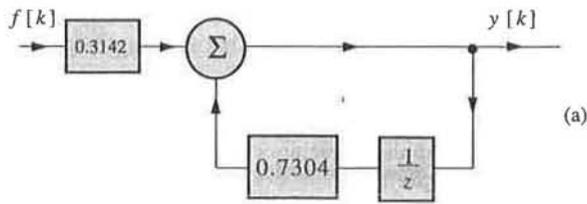


Fig. 12.11 An example of filter design by the impulse invariance method: (a) filter realization (b) amplitude response (c) phase response.

Also, according to Eq. (12.47) (with $\omega_c = 10^5$)

$$H_a(j\omega) = \frac{\omega_c}{j\omega + \omega_c} = \frac{10^5}{j\omega + 10^5} \tag{12.52}$$

Therefore

$$|H_a(j\omega)| = \frac{10^5}{\sqrt{\omega^2 + 10^{10}}} \quad \text{and} \quad \angle H_a(j\omega) = -\tan^{-1} \frac{\omega}{10^5}$$

Figures 12.11b and 12.11c show the amplitude and phase response of the analog and the (normalized)† digital filter over the frequency range $0 \leq \omega \leq \pi/T = 10^6$. Observe that

†The frequency response $H[e^{j\omega T}]$ is higher than the desired frequency response $H_a(j\omega)$ because of aliasing. We can partially correct this difference by multiplying $H[z]$ with a normalizing constant, forcing the resulting $H[e^{j\omega T}]$ to be equal to $H_a(j\omega)$ at $\omega = 0$. The normalizing constant K is defined as the ratio of $H_a(j0)$ to $H[e^{j0}] = H[1]$, which in this case is $1/1.653 = 0.858$. The normalized amplitude response in Fig. 12.11b is that of $0.858 \left(\frac{0.1\pi z}{z - 0.7304} \right)$.

the behavior of the analog and the digital filter is very close over the range $\omega \leq \omega_c = 10^5$. However, for higher frequencies, there is considerable aliasing, especially in the phase spectrum. ■

⊙ **Computer Example C12.3**

Using MATLAB, find the impulse invariance digital filter to realize the first-order analog Butterworth filter in Example 12.4.

The analog filter transfer function is $10^5/(s + 10^5)$ and the sampling interval $T = 10^{-6}\pi$. A suitable MATLAB function to solve this problem is 'impinvar'. The input data are the coefficients of the numerator and the denominator polynomials of $H_a(s)$ [entered as $(n + 1)$ -element vectors num and den] and the sampling interval T. MATLAB returns b and a, the numerator and the denominator polynomial coefficients of the desired digital filter $H[z]$.

In designing impulse invariance filter, we use the criterion $h[k] = Th_a(kT)$ in Eq. (12.39), whereas most books, including MATLAB, use the criterion $h[k] = h_a(kT)$. Hence, our answer will be T times the answer returned by MATLAB. To correct this discrepancy, we multiply num by T .

```
T=pi/10^6; num=T*[0 10^5]; den=[1 10^5];
Fs=1/T;
[b,a]=impinvar(num,den,Fs)
```

MATLAB returns $b=0.3142$ and $a=1 - 0.7304$. Therefore

$$H[z] = \frac{0.3142z}{z - 0.7304}$$

a conclusion which agrees with our result in Eq. (12.50). To plot the amplitude and the phase response, we can use the last 8 functions in Example C12.1. ⊙

△ **Exercise E12.4**

Design a digital filter to realize an analog transfer function

$$H_a(s) = \frac{20}{s + 20}$$

Answer: $H[z] = \frac{20Tz}{z - e^{-20T}}$ with $T = \frac{\pi}{2000}$ ▽

Limitations of the Impulse Invariance Method

The impulse invariance method is handicapped by aliasing. Consequently this method can be used to design filters where $H_a(j\omega)$ becomes negligible beyond some frequency B Hz. This condition restricts the procedure to lowpass and bandpass filters. The impulse invariance method cannot be used for highpass or bandstop filters. Moreover, to reduce aliasing effects the sampling rate has to be very high, which makes its implementation costly. In general, the frequency-domain method discussed in the next section is superior to this method.

12.6 Recursive Filter Design by the Frequency-Domain Criterion: The Bilinear Transformation Method

The bilinear transformation method discussed in this section is preferable to the impulse invariance method in filtering problems where the gains are constant over certain bands (piecewise constant amplitude response). This condition exists in lowpass, bandpass, highpass, and bandstop filters. Moreover, this method requires

a lower sampling rate compared to the impulse invariance method because of the absence of aliasing. In addition, the filter rolloff characteristics are sharper with this method compared to those obtained using the impulse invariance method. The absence of aliasing is the result of one-to-one mapping from s plane to z plane inherent in this method.

The frequency-domain design criterion is [see Eq. (12.37)]

$$\lim_{T \rightarrow 0} H[e^{sT}] = H_a(s) \quad (12.53)$$

Let us consider the following power series for the hyperbolic tangent (see Sec. B.7-3)

$$\tanh\left(\frac{sT}{2}\right) \equiv \frac{e^{sT/2} - e^{-sT/2}}{e^{sT/2} + e^{-sT/2}} = \left[\frac{sT}{2} - \frac{1}{3} \left(\frac{sT}{2}\right)^3 + \frac{2}{15} \left(\frac{sT}{2}\right)^5 + \dots \right] \quad (12.54)$$

For small T ($T \rightarrow 0$), we can ignore the higher-order terms in the infinite series on the right-hand side to yield

$$\lim_{T \rightarrow 0} \left(\frac{e^{sT/2} - e^{-sT/2}}{e^{sT/2} + e^{-sT/2}} \right) = \frac{sT}{2}$$

Therefore, as $T \rightarrow 0$

$$\begin{aligned} s &= \left(\frac{2}{T}\right) \frac{e^{sT/2} - e^{-sT/2}}{e^{sT/2} + e^{-sT/2}} \\ &= \left(\frac{2}{T}\right) \frac{e^{sT} - 1}{e^{sT} + 1} \end{aligned}$$

Equation (12.53) now can be expressed as

$$H[e^{sT}] = H_a\left(\frac{2}{T} \frac{e^{sT} - 1}{e^{sT} + 1}\right) \quad (12.55a)$$

From this result, it follows that

$$H[z] = H_a\left(\frac{2}{T} \frac{z - 1}{z + 1}\right) \quad (12.55b)$$

$$= H_a(s) \Big|_{s = \frac{2}{T} \frac{z-1}{z+1}} \quad (12.55c)$$

Therefore, we can obtain $H[z]$ from $H_a(s)$ by using the transformation†

$$s = \left(\frac{2}{T}\right) \frac{z - 1}{z + 1} \quad (12.57)$$

This transformation is known as the **bilinear transformation**.

†There exist other transformations, which can be used to derive $H[z]$ from $H_a(s)$. We start with the power series

$$e^{-sT} = 1 - sT + \frac{1}{2}(sT)^2 - \frac{1}{3}(sT)^3 + \dots$$

In the limit as $T \rightarrow 0$, all but the first two terms on the right-hand side can be ignored. This yields

$$s = \frac{1}{T}(1 - e^{-sT})$$

This results in a transformation

Choice of T in Bilinear Transformation Method

Because of the absence of aliasing in the bilinear transformation method, the value of the sampling interval T can be much smaller compared to the impulse invariance method. By absence of aliasing we mean only the kind of aliasing observed in impulse invariance method (Fig. 12.10b). The signal aliasing, which limits the highest usable frequency, is still present. Thus if the highest frequency to be processed is \mathcal{F}_h Hz, then to avoid signal aliasing, we must use [see Eq. (8.17c)]

$$T \leq \frac{1}{2\mathcal{F}_h} \quad (12.58)$$

■ Example 12.5

Using the bilinear transformation, synthesize

$$H_a(s) = \frac{\omega_c}{s + \omega_c} \quad \omega_c = 10^5$$

From Eq. (12.57), we obtain

$$\begin{aligned} H[z] &= \frac{\omega_c}{\left(\frac{2}{T} \frac{z-1}{z+1}\right) + \omega_c} \\ &= \frac{\omega_c T (z+1)}{(2 + \omega_c T)z - (2 - \omega_c T)} \end{aligned}$$

We should use Eq. (12.58) to select a suitable value for T . However, to facilitate comparison with the impulse invariance method, we choose here the same value for T as that in Example 12.4: $T = \frac{\pi}{10\omega_c}$. The substitution of $\omega_c T = \pi/10$ in the above equation yields

$$H[z] = 0.1357 \left(\frac{z+1}{z-0.7284} \right)$$

Hence

$$\begin{aligned} H[e^{j\omega T}] &= \frac{0.1357(e^{j\omega T} + 1)}{e^{j\omega T} - 0.7284} \\ &= \frac{0.1357(\cos \omega T + 1 + j \sin \omega T)}{\cos \omega T - 0.7284 + j \sin \omega T} \quad T = \pi \times 10^{-6} \end{aligned}$$

From this we obtain

$$|H[e^{j\omega T}]| = \left[\frac{0.024(1 + \cos \omega T)}{1 - 0.9518 \cos \omega T} \right]^{1/2} \quad T = \pi \times 10^{-6} \quad (12.59a)$$

and

$$s = \frac{1}{T} \left(1 - \frac{1}{z} \right) = \frac{z-1}{Tz} \quad (12.56a)$$

Similarly, starting with the power series for e^{sT} yields the transformation

$$s = \frac{1}{T} (z-1) \quad (12.56b)$$

These are strikingly simple transformations, which work reasonably well for lowpass and bandpass filters with low resonant frequencies. They cannot be used for highpass and bandstop filters, however, and they are inferior to bilinear transformation. The transformation in Eq. (12.56b) also has a stability problem.

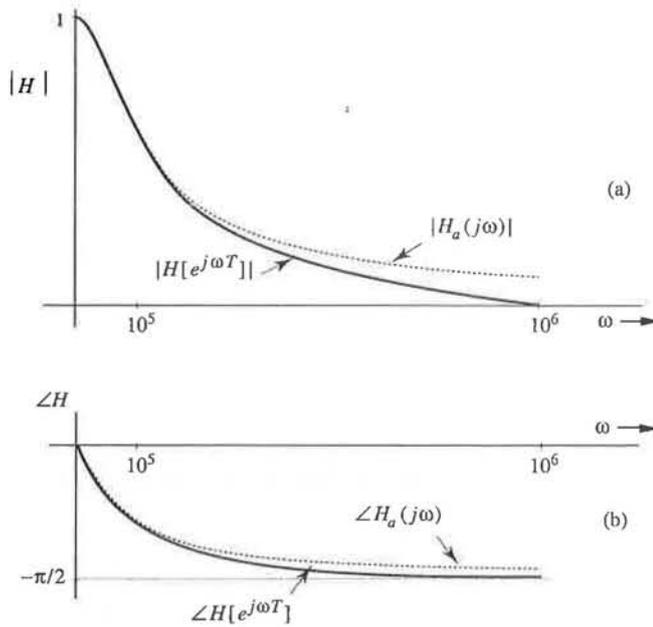


Fig. 12.12 Bilinear transformation method of design: (a) amplitude response (b) phase response.

$$\angle H[e^{j\omega T}] = \tan^{-1} \frac{\sin \omega T}{1 + \cos \omega T} - \tan^{-1} \frac{\sin \omega T}{\cos \omega T - 0.7285} \quad (12.59b)$$

Figure 12.12 shows $|H|$ and $\angle H$ as computed from Eqs. (12.59). Compare these with the filter characteristics obtained from the impulse invariant method (Fig. 12.11). ■

⊙ Computer Example C12.4

Using MATLAB, find the bilinear transformed digital filter to realize the first-order analog Butterworth filter in Example 12.5.

The analog filter transfer function is $10^5/(s + 10^5)$ and the sampling interval $T = 10^{-6}\pi$. Hence, the sampling frequency $F_s = 10^6/\pi$. A suitable MATLAB function to solve this problem is 'bilinear'. The input data are the coefficients of the numerator and the denominator polynomials of $H_a(s)$ [entered as $(n + 1)$ -element vectors num and den] and the sampling frequency F_s Hz. MATLAB returns b and a, the numerator and the denominator polynomial coefficients of the desired digital filter $H[z]$.

```
Fs=10^6/pi; num=[0 10^5];den=[1 10^5];
[b,a]=bilinear(num,den,Fs)
```

MATLAB returns $b=0.1358 \ 0.1358$ and $a=1 \ -0.7285$. Therefore

$$H[z] = \frac{0.1358(z + 1)}{z - 0.7285}$$

which agrees with the answer found in Example 12.5. To plot the amplitude and the phase response, we can use the last 8 functions in Example C12.1. ⊙

Frequency Prewarping Inherent in Bilinear Transformation

Figure 12.12 shows that $|H[e^{j\omega T}]| \simeq |H_a(j\omega)|$ for small ω . For large values of ω , the error increases. Moreover, $|H[e^{j\omega T}]| = 0$ at $\omega = \pi/T$. In fact, it appears as if the

entire frequency band (0 to ∞) in $|H_a(j\omega)|$ is compressed within the range $(0, \frac{\pi}{T})$ in $H[e^{j\omega T}]$. Such warping of the frequency scale is peculiar to this transformation. To understand this behavior, consider Eq. (12.55a) with $s = j\omega$

$$H[e^{j\omega T}] = H_a\left(\frac{2}{T} \frac{e^{j\omega T} - 1}{e^{j\omega T} + 1}\right) = H_a\left(\frac{2}{T} \frac{e^{\frac{j\omega T}{2}} - e^{-\frac{j\omega T}{2}}}{e^{\frac{j\omega T}{2}} + e^{-\frac{j\omega T}{2}}}\right) = H_a\left(j \frac{2}{T} \tan \frac{\omega T}{2}\right)$$

Therefore, response of the resulting digital filter at some frequency ω_d is

$$\begin{aligned} H[e^{j\omega_d T}] &= H_a\left(j \frac{2}{T} \tan \frac{\omega_d T}{2}\right) \\ &= H_a(j\omega_a) \end{aligned} \quad (12.60)$$

where

$$\omega_a = \frac{2}{T} \tan \frac{\omega_d T}{2} \quad (12.61a)$$

Thus, in the resulting digital filter, the behavior of the desired response $H_a(j\omega)$ at some frequency ω_a appears not at ω_a but at frequency ω_d , where [from Eq. (12.61a)]

$$\omega_d = \frac{2}{T} \tan^{-1} \frac{\omega_a T}{2} \quad (12.61b)$$

Figure 12.13a shows the plot of ω_d as a function of ω_a . For small ω_a , the curve in Fig. 12.13a is practically linear, so $\omega_d \simeq \omega_a$. At higher values of ω_a , there is considerable diversion in the values of ω_a and ω_d . Thus, the digital filter imitates the desired analog filter at low frequencies, but at higher frequencies there is considerable distortion. Using this method, if we are trying to synthesize a filter to realize $H_a(j\omega)$ depicted in Fig. 12.13b, the resulting digital filter frequency response will be, as illustrated in Fig. 12.13c. The analog filter behavior in the entire range of ω_a from 0 to ∞ is compressed in the digital filter in the range of ω_d from 0 to π/T . This is as if a promising 20 year old man, who, after learning that he has only a year to live, tries to crowd his last year with every possible adventure, passion, and sensation that a normal human being would have experienced in an entire lifetime. This compression and frequency warping effect is the peculiarity of the bilinear transformation.

There are two ways of overcoming frequency warping. The first is to reduce T (increase the sampling rate) so that the signal bandwidth is kept well below $\frac{\pi}{T}$ and $\omega_a \simeq \omega_d$ over the desired frequency band. This step is easy to execute, but it requires a higher sampling rate (lower T) than necessary. The second approach, known as **prewarping**, solves the problem without unduly reducing T .

12.6-1 Bilinear Transformation Method with Prewarping

In prewarping, we start not with the desired $H_a(j\omega)$ but with a prewarped $H_a(j\omega)$ in such a way that the warping because of bilinear transformation will compensate for the prewarping exactly. The idea here is to begin with a distorted analog filter (prewarping) so that the distortion caused by bilinear transformation will be canceled by the built-in (prewarping) distortion. The idea is similar to

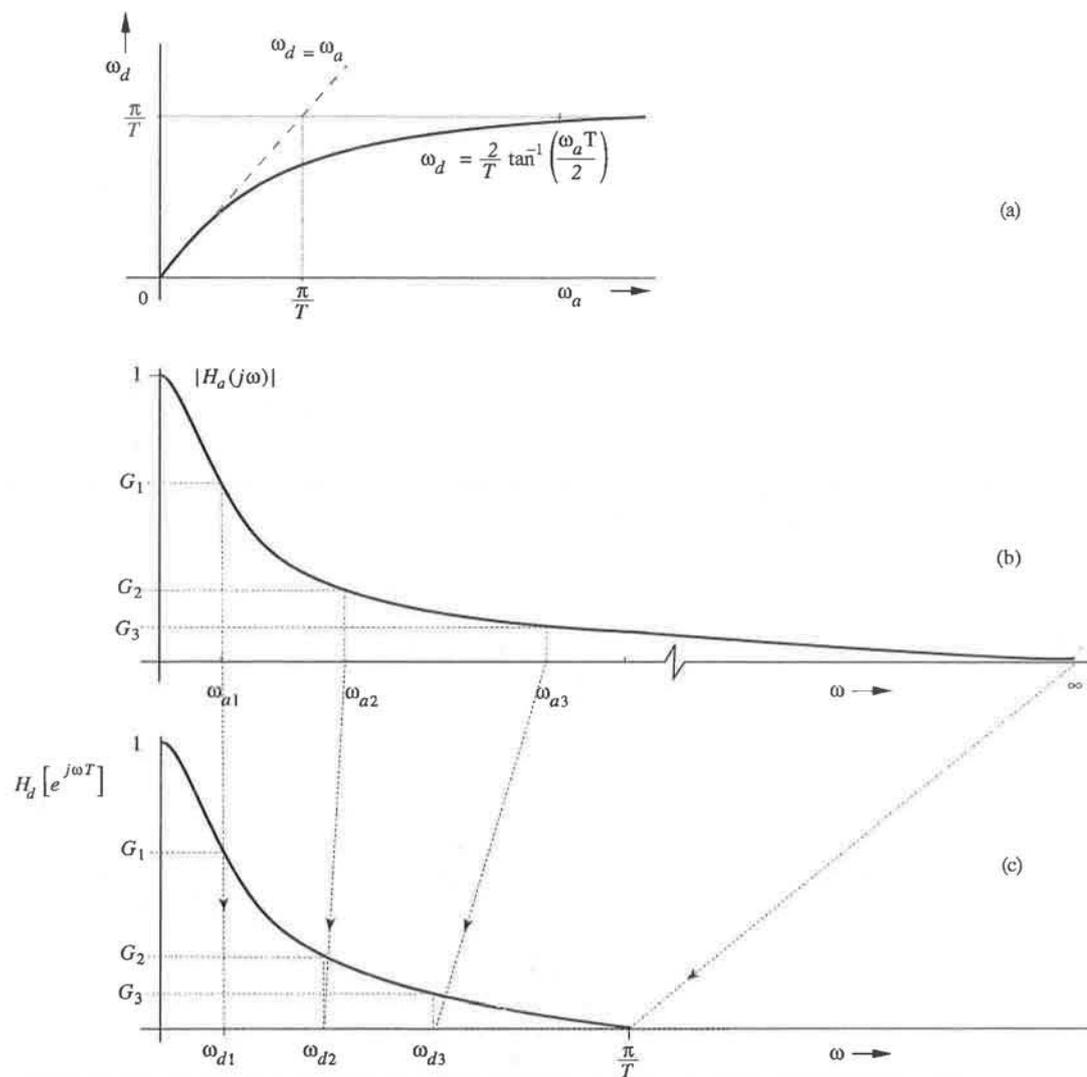


Fig. 12.13 Frequency warping in bilinear transformation: (a) mapping relationship of analog and digital frequencies (b) analog response (c) corresponding digital response.

the one used in prestressed concrete, in which a concrete beam is precompressed initially. When loaded, the beam experiences tension, which is canceled by the built-in compression.

Usually the prewarping is done at certain critical frequencies rather than over the entire band. The final filter behavior is exactly equal to the desired behavior at these selected frequencies. Such a filter is adequate for most filtering problems if we choose the critical frequencies properly.

If we require a filter to have gains g_1, g_2, \dots, g_m at frequencies (critical frequencies) $\omega_1, \omega_2, \dots, \omega_m$ respectively, then we must start with an analog filter $H'(j\omega)$ which has gains g_1, g_2, \dots, g_m at frequencies $\omega_1', \omega_2', \dots, \omega_m'$ respectively, where

$$\omega_i' = \frac{2}{T} \tan \frac{\omega_i T}{2} \quad i = 1, 2, \dots, m \quad (12.62)$$

This results in prewarped filter $H'(j\omega)$. Application of the bilinear transformation (12.57) to this filter yields the desired digital filter which has gains g_1, g_2, \dots, g_m at frequencies $\omega_1, \omega_2, \dots, \omega_m$ respectively. This is because, according to Eq. (12.61a), the behavior of the analog filter at a frequency ω_i' appears in the digital filter at frequency

$$\frac{2}{T} \tan^{-1} \left(\frac{\omega_i' T}{2} \right) = \frac{2}{T} \tan^{-1} \left[\tan \left(\frac{\omega_i T}{2} \right) \right] = \omega_i$$

We clarify these ideas with an example of a lowpass Butterworth filter.

■ **Example 12.6**

Design a lowpass filter with the following specifications: The gain of unity at $\omega = 0$, and the gain is to be no less than -2 dB ($G_p = 0.785$) over the passband $0 \leq \omega \leq 10$. The gain is to be no greater than -11 dB ($G_s = 0.2818$) over the stopband $\omega \geq 15$. The highest frequency to be processed is $\omega_h = 35$ rad/s, which yields $T \leq \pi/35$. Let us use $T = \pi/35$.

The specifications for a Butterworth filter for this design are $\omega_p = 8$, $\omega_s = 15$, $\hat{G}_p = -2$ dB, and $\hat{G}_s = -11$ dB. In the first step, we prewarp the critical frequencies ω_p and ω_s according to Eq. (12.62):

$$\begin{aligned} \omega_p' &= \frac{2}{T} \tan \frac{\omega_p T}{2} = \frac{70}{\pi} \tan \left(\frac{4\pi}{35} \right) = 8.3623 \\ \omega_s' &= \frac{2}{T} \tan \frac{\omega_s T}{2} = \frac{70}{\pi} \tan \left(\frac{15\pi}{70} \right) = 17.7696 \end{aligned}$$

In the second step, we design a Butterworth filter with critical frequencies $\omega_p' = 8.3623$ and $\omega_s' = 17.7696$ with $\hat{G}_p = -2$ dB and $\hat{G}_s = -11$ dB. The value of n (order of the filter) is obtained from Eq. (7.39):

$$n = \frac{\log[(10^{1.1} - 1)/(10^{0.2} - 1)]}{2 \log(17.7696/8.3623)} = 1.9405$$

We round up the value of n to 2. There are two possible values of ω_c' . We shall choose the one given by equation (7.41), which satisfies the stopband specifications exactly, but oversatisfies that in the passband. This choice yields the 3-dB cutoff frequency ω_c' as

$$\omega_c' = \frac{17.7696}{(10^{1.1} - 1)^{\frac{1}{4}}} = 9.6308$$

From Table 7.1, we find the prewarped filter transfer function $H_a(s)$ for $n = 2$ and $\omega_c' = 9.6308$ as

$$H_a(s) = \frac{1}{\left(\frac{s}{9.6308}\right)^2 + \sqrt{2} \left(\frac{s}{9.6308}\right) + 1} = \frac{92.7529}{s^2 + 13.62s + 92.7529}$$

Finally, we obtain $H[z]$ from $H_a(s)$, using the bilinear transformation

$$s = \left(\frac{2}{T}\right) \frac{z-1}{z+1} = \left(\frac{70}{\pi}\right) \frac{z-1}{z+1}$$

This substitution yields

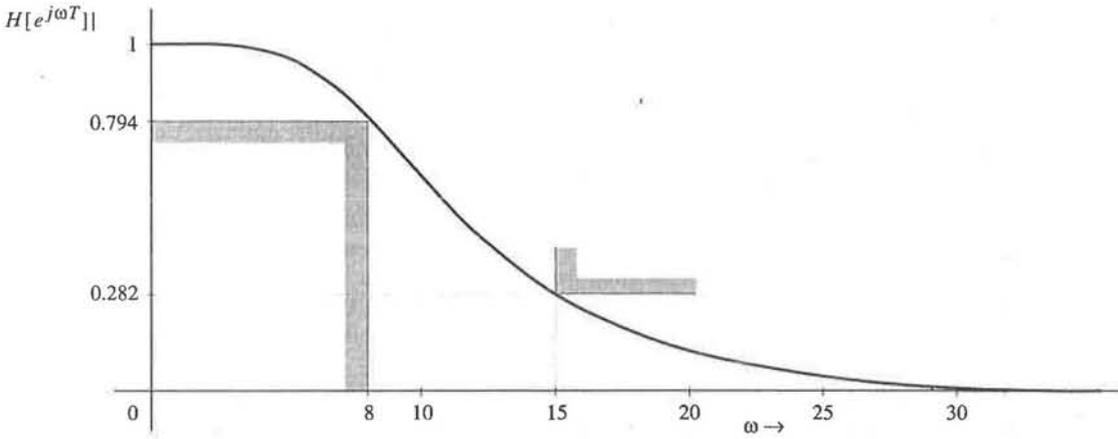


Fig. 12.14 Amplitude response of the filter in Example 12.6.

$$\begin{aligned}
 H[z] &= H_a(s) \Big|_{s=\frac{z-1}{z+1}} \\
 &= \frac{92.7529}{s^2 + 13.62s + 92.7529} \Big|_{s=\frac{z-1}{z+1}} \\
 &= \frac{0.1039(z+1)^2}{z^2 - 0.9045z + 0.3201}
 \end{aligned}$$

The frequency response of this filter is given by

$$H[e^{j\omega T}] = \frac{0.1039(e^{j\omega T} + 1)^2}{e^{j2\omega T} - 0.9045e^{j\omega T} + 0.3201}$$

The amplitude response $|H[e^{j\omega T}]|$, with $T = 70/\pi$, is depicted in Fig. 12.14. ■

Summary of the Bilinear Transformation Method with Prewarping

In the bilinear transformation method with prewarping, all the critical frequencies ω_i are transformed (prewarped) by the equation

$$\omega_i' = \frac{2}{T} \tan \frac{\omega_i T}{2} \quad i = 1, 2, \dots, m \quad (12.63a)$$

The prewarped cutoff frequency ω_c' , determined by using prewarped critical frequencies, is used to find the prewarped analog filter transfer function $H_a(s)$. Finally, we replace s with $\frac{2}{T} \frac{z-1}{z+1}$ in $H_a(s)$ to obtain the desired digital filter transfer function $H[z]$

$$H[z] = H_a(s) \Big|_{s=\frac{2}{T} \frac{z-1}{z+1}} \quad (12.63b)$$

A Simplified Procedure

The above procedure can be simplified by observing that the scaling factor $\frac{2}{T}$ is irrelevant in this manipulation and can be ignored. Instead of using Eqs. (12.63a) and (12.63b), we can use the simplified equations

$$\omega_i' = \tan \frac{\omega_i T}{2} \quad i = 1, 2, \dots, m \quad (12.64a)$$

and

$$s = \frac{z-1}{z+1} \quad (12.64b)$$

This simplification works because the factor $2/T$ in Eq. (12.63a) is a frequency scaling factor, and ignoring it in Eq. (12.64a) results in the pretransformed filter that is scaled by a factor $2/T$ in the frequency scale. This scaling is undone by using Eq. (12.64b) instead of Eq. (12.63b).

To demonstrate the procedure, we shall redo Example 12.6 using this simplification. In the first step, we prewarp the critical frequencies ω_p and ω_s according to Eq. (12.64a):

$$\omega_p' = \tan \frac{\omega_p T}{2} = \tan \left(\frac{4\pi}{35} \right) = 0.3753$$

$$\omega_s' = \tan \frac{\omega_s T}{2} = \tan \left(\frac{15\pi}{70} \right) = 0.7975$$

In the second step, we design a Butterworth filter with critical frequencies $\omega_p' = 0.3753$ and $\omega_s' = 0.7975$ with $\hat{G}_p = -2$ dB and $\hat{G}_s = -11$ dB. The value of n (order of the filter) is found from Eq. (7.39):

$$n = \frac{\log[(10^{1.1} - 1)/(10^{0.2} - 1)]}{2 \log(0.7975/0.3753)} = 1.9405$$

We round up the value of n to 2. Also from Eq. (7.41), we find the 3-dB cutoff frequency ω_c' as

$$\omega_c' = \frac{0.7975}{(10^{1.1} - 1)^{\frac{1}{4}}} = 0.4322$$

From Table 7.1, we find the prewarped filter transfer function $H_a(s)$ for $n = 2$ and $\omega_c' = 0.4322$ as

$$H_a(s) = \frac{1}{\left(\frac{s}{0.4322}\right)^2 + \sqrt{2}\left(\frac{s}{0.4322}\right) + 1} = \frac{0.1868}{s^2 + 0.6112s + 0.1868}$$

Finally, we obtain $H[z]$ from $H_a(s)$, using the simplified bilinear transformation in Eq. (12.64b):

$$s = \frac{z-1}{z+1}$$

Therefore

$$H[z] = H_a(s) \Big|_{s=\frac{z-1}{z+1}} = \frac{0.1868}{s^2 + 0.6112s + 0.1868} \Big|_{s=\frac{z-1}{z+1}} = \frac{0.1039(z+1)^2}{z^2 + 0.904z + 0.3201}$$

which is identical to the result obtained earlier.

⊙ **Computer Example C12.5**

Design a lowpass digital filter to meet the specifications in Example 12.6, using bilinear transformation with prewarping.

We shall give here MATLAB functions to design the four types of approximations: Butterworth, Chebyshev, inverse Chebyshev, and elliptic. The input data asks for frequencies so normalized that the sampling radian frequency is 2. This requirement means the sampling radian frequency, which is $2\pi/T$, must be normalized to 2. Therefore, all the radian frequencies can be normalized by multiplying each of them by T/π . In the present case, $T = \pi/35$ so that $T/\pi = 1/35$. Thus, to normalize any radian frequency, we just divide it by 35. The normalized W_p and W_s are $8/35$ and $15/35$, respectively.

```
Wp=8/35;Ws=15/35;Gp=-2;Gs=-11;T=pi/35;
```

```
% Butterworth
```

```
[n,Wn]=buttord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=butter(n,Wn)
```

```
% Chebyshev
```

```
[n,Wn]=cheb1ord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=cheby1(n,-Gp,Wn)
```

```
% Inverse Chebyshev
```

```
[n,Wn]=cheb2ord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=cheby2(n,-Gs,Wn)
```

```
% Elliptic
```

```
[n,Wn]=ellipord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=ellip(n,-Gp,-Gs,Wn)
```

```
% Plotting Amplitude and Phase Response
```

```
W=0:.001:pi;W=W';
```

```
H=freqz(b,a,W);
```

```
w=W/T;
```

```
mag=abs(H);
```

```
phase=180/pi*unwrap(angle(H));
```

```
subplot(2,1,1);
```

```
plot(w,mag);grid;
```

```
subplot(2,1,2);
```

```
plot(w,phase);grid
```

MATLAB returns $b=0.1039 \ 0.2078 \ 0.1039$ and $a=1 \ -0.9045 \ 0.3201$ for Butterworth option. Therefore

$$H[z] = \frac{0.1039(z+1)^2}{z^2 - 0.9045z + 0.3201}$$

a result, which agrees with the answer found in Example 12.7. ⊙

△ **Exercise E12.5**

Design a first-order lowpass Butterworth filter using the prewarping method so that the analog and digital gains are identical at $\omega = 0$ and at the 3-dB cutoff frequency ω_c . Use $T = \pi/4\omega_c$. Answer:

$$H[z] = \frac{0.8284(z+1)}{2.8284z - 1.1716} = \frac{0.2929(z+1)}{z - 0.4142}$$

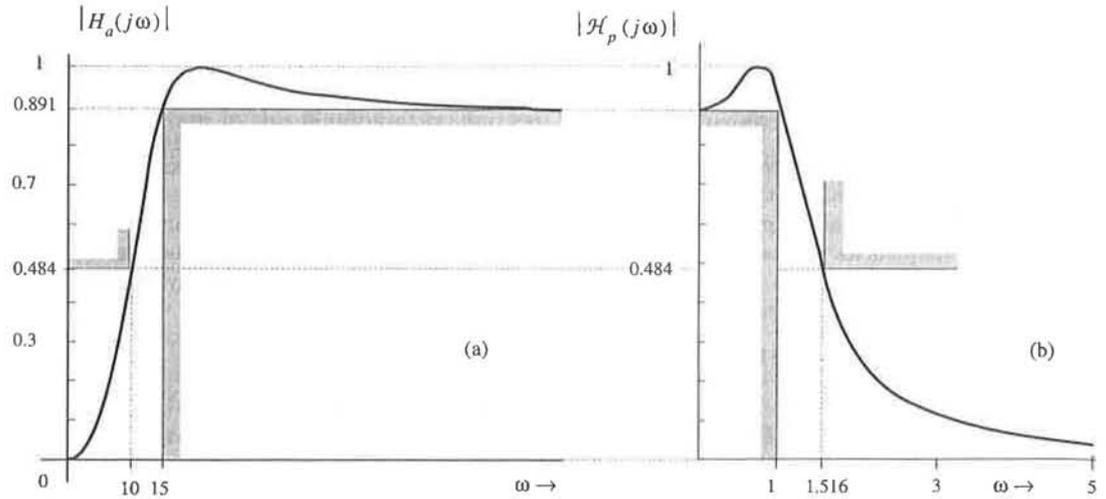


Fig. 12.15 Chebyshev highpass filter design using bilinear transformation with prewarping method.

▽

■ Example 12.7: Highpass Filter Design

Design a 1-dB ripple Chebyshev highpass filter with the following specifications (depicted by the brick walls in Fig. 12.15a): The stopband gain $\hat{G}_s \leq -6.3$ dB ($G_s \leq 0.484$) over the stopband $0 \leq \omega \leq 10$ ($\omega_s = 10$). The ripple $\hat{r} \leq 1$ dB ($G_p \geq 0.891$) over a passband $\omega \geq 15$ ($\omega_p = 15$). The highest frequency to be processed is $\omega_h = 80$ radians/s.

In order to select a suitable value of T , we use Eq. (12.58) to avoid signal aliasing:

$$T \leq \frac{\pi}{\omega_h} = \frac{\pi}{80} \implies \text{Let us choose } T = \frac{\pi}{100}$$

The critical frequencies are $\omega_s = 10$ and $\omega_p = 15$. The prewarped critical frequencies, according to Eq. (12.64a), are

$$\omega_s' = \tan \frac{\omega_s T}{2} = \tan \left(\frac{\pi}{20} \right) = 0.1584$$

$$\omega_p' = \tan \frac{\omega_p T}{2} = \tan \left(\frac{3\pi}{40} \right) = 0.24$$

In the second step, we design a prewarped Chebyshev highpass filter with critical frequencies $\omega_s' = 0.1584$ and $\omega_p' = 0.24$ with $\hat{r} = -1$ dB and $\hat{G}_s = -6.3$ dB (Fig. 12.15b). Following the procedure in Sec. 7.6-1, we first design a prototype lowpass filter with specifications, as indicated in Fig. 12.15b. Observe that the critical frequencies of the prototype filter are 1 (passband) and $\omega_p'/\omega_s' = 1.515$ (stopband) as explained in Sec. 7.6-1

The value of n needed to satisfy these specifications is given by [see Eq. (7.49b)]

$$n = \frac{1}{\cosh^{-1} \left(\frac{1.515}{1} \right)} \cosh^{-1} \left[\frac{(10^{0.63} - 1)}{10^{0.1} - 1} \right]^{1/2} = 1.988$$

We round up the value of n to 2. From Table 7.4 (Chebyshev filter with $\hat{r} = 1$ and $n = 2$) we obtain the following prototype transfer function

$$\mathcal{H}_p(s) = \frac{0.982614}{s^2 + 1.0978s + 1.1025} \quad (12.65)$$

Next, to obtain the desired highpass transfer function, we replace s with ω_p'/s in the above prototype transfer function [see Eq. (7.55)]. To obtain the desired digital transfer function $H[z]$, we then replace s with $\frac{z-1}{z+1}$ [the bilinear transformation in Eq. (12.64b)]. This two-step operation may be combined in a single-step transformation as

$$s \Rightarrow \frac{\omega_p'}{\left(\frac{z-1}{z+1}\right)} = \frac{\omega_p'(z+1)}{(z-1)} \quad (12.66)$$

In this case $\omega_p' = 0.24$ so that we replace s with $\frac{0.24(z+1)}{z-1}$ in the prototype transfer function in Eq. (12.65) to obtain the desired digital transfer function

$$H[z] = \frac{0.6902(z-1)^2}{z^2 - 1.4678z + 0.6298}$$

The continuous curve in Fig. 12.15a shows the amplitude response of this filter. ■

⊙ Computer Example C12.6

Design a highpass digital filter to meet the specifications in Example 12.7, using bilinear transformation with prewarping.

As before, we shall give here MATLAB functions to design the four basic types of approximations. The input data asks for frequencies so normalized that the sampling radian frequency is 2. As explained in Example C12.5, all the radian frequencies can be normalized by multiplying each of them by T/π . In the present case, $T = \pi/100$ so that $T/\pi = 1/100$. Thus, to normalize any radian frequency, we just divide it by 100. The normalized ω_p and ω_s are 15/100 and 10/100, respectively.

```
Wp=0.15;Ws=0.1;-Gp=1;-Gs=6.3;T=pi/100;
```

```
% Butterworth
```

```
[n,Wn]=Buttord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=butter(n,Wn,'high');
```

```
% Chebyshev
```

```
[n,Wn]=cheblord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=cheby1(n,-Gp,Wn,'high');
```

```
% Inverse Chebyshev
```

```
[n,Wn]=cheb2ord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=cheby2(n,-Gs,Wn,'high')
```

```
% Elliptic
```

```
[n,Wn]=ellipord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=ellip(n,-Gp,-Gs,Wn,'high')
```

MATLAB returns $b=0.6902 \ -1.3804 \ 0.6902$ and $a=1 \ -1.4678 \ 0.6298$ for Chebyshev option. Therefore

$$H[z] = \frac{0.6902(z-1)^2}{z^2 - 1.4678z + 0.6298}$$

which agrees with the answer found in Example 12.6. To plot the amplitude and the phase response, we can use the last 9 functions in Example C12.5. ⊙

Bandpass and Bandstop Filters

For bandpass and bandstop filters, we follow a similar procedure. All the critical frequencies are first prewarped using the simplified form in Eq. (12.64a). Next, we determine a prototype lowpass filter, which is then converted to the desired analog filter using appropriate transformations discussed in Sec. 7.6. Finally, we use the bilinear transformation in Eq. (12.64b) to obtain the desired digital filter. As in the case of the highpass filter (discussed in Example 12.7), we can combine the two transformations into a single transformation. For the bandpass filter, we first use the transformation

$$T(s) = \frac{s^2 + \omega_{p_1}' \omega_{p_2}'}{(\omega_{p_2}' - \omega_{p_1}')s}$$

and then use the bilinear transformation in Eq. (12.64b). Thus, in the first step we replace s in the prototype transfer function $\mathcal{H}_p(s)$ with $T(s)$ (the frequency transformation). In the second step we replace s with $\frac{z-1}{z+1}$ (the simplified bilinear transformation). Thus, the final transformation is equivalent to replacing s with $T_{bp}[z]$ in the prototype filter transfer function $\mathcal{H}_p(s)$, where

$$\begin{aligned} T_{bp}[z] &= T(s) \Big|_{s=\frac{z-1}{z+1}} = \frac{s^2 + \omega_{p_1}' \omega_{p_2}'}{(\omega_{p_2}' - \omega_{p_1}')s} \Big|_{s=\frac{z-1}{z+1}} \\ &= \frac{(z-1)^2 + \omega_{p_1}' \omega_{p_2}'(z+1)^2}{(\omega_{p_2}' - \omega_{p_1}')(z^2 - 1)} \\ &= \frac{(\omega_{p_1}' \omega_{p_2}' + 1)z^2 + 2(\omega_{p_1}' \omega_{p_2}' - 1)z + (\omega_{p_1}' \omega_{p_2}' + 1)}{(\omega_{p_2}' - \omega_{p_1}')(z^2 - 1)} \end{aligned}$$

Using the same argument, we can show that for the bandstop filter, the desired digital filter transfer function $H[z]$ can be obtained from the corresponding bandstop prototype filter $\mathcal{H}_p(s)$ by replacing s with $T_{bs}[z]$, which is the reciprocal of $T_{bp}[z]$. Both these transformations can be expressed in a more compact form as

$$T_{bp}[z] = \frac{z^2 + 2az + 1}{b(z^2 - 1)} \quad (12.67a)$$

$$T_{bs}[z] = \frac{b(z^2 - 1)}{z^2 + 2az + 1} \quad (12.67b)$$

where

$$a = \frac{\omega_{p_1}' \omega_{p_2}' - 1}{\omega_{p_1}' \omega_{p_2}' + 1} \quad \text{and} \quad b = \frac{\omega_{p_2}' - \omega_{p_1}'}{\omega_{p_1}' \omega_{p_2}' + 1} \quad (12.68)$$

Thus, a digital filter transfer function $H[z]$ can be obtained from $\mathcal{H}_p(s)$ by replacing s with $T_{bp}[z]$ for the bandpass filter, and replacing s with $T_{bs}[z]$ for the bandstop filter.

Example 12.8

Design a digital Butterworth bandpass filter with amplitude response specifications illustrated by the brick walls in Fig. 12.16a with $\omega_{p1} = 1000$, $\omega_{p2} = 2000$, $\omega_{s1} = 450$, $\omega_{s2} = 4000$, $G_p = 0.7852$ (-2.1 dB), and $G_s = 0.1$ (-20 dB). Take $T = \pi/10,000$.

The solution is executed in 3 steps: In the first step, we determine the prewarped critical frequencies. In the second step, the lowpass prototype filter transfer function $\mathcal{H}_p(s)$ is found from the prewarped critical frequencies. Finally, the desired $H[z]$ is found from $\mathcal{H}_p(s)$ using the lowpass analog to bandpass digital transformation by replacing s in $\mathcal{H}_p(s)$ with $T_{bp}[z]$.

Step 1: Find prewarped critical frequencies

The prewarped frequencies ω_{p1}' , ω_{p2}' , ω_{s1}' , and ω_{s2}' corresponding to the four critical frequencies ω_{p1} , ω_{p2} , ω_{s1} , and ω_{s2} using Eq. (12.64a):

$$\begin{aligned}\omega_{s1}' &= \tan \frac{\omega_{s1}T}{2} \\ &= \tan \left(\frac{450\pi}{20,000} \right) = 0.0708 \\ \omega_{p1}' &= \tan \left(\frac{1000\pi}{20,000} \right) = 0.1584 \\ \omega_{p2}' &= \tan \left(\frac{2000\pi}{20,000} \right) = 0.3249 \\ \omega_{s2}' &= \tan \left(\frac{4000\pi}{20,000} \right) = 0.7265\end{aligned}$$

Step 2: Find $\mathcal{H}_p(s)$, the prewarped lowpass prototype analog filter

This procedure with 5 substeps is identical to step 1 in the design of an analog bandpass filter discussed in Example 7.10 (Sec. 7.6-2). The 5 substeps are:

Step 2.1: Find ω_s' for the prototype filter. For the prototype lowpass filter transfer function $\mathcal{H}_p(s)$ with amplitude response, as depicted in Fig. 12.16b. The frequency ω_s' is found [using Eq. (7.56)] to be the smaller of

$$\frac{(0.1584)(0.3249) - (0.0708)^2}{0.0708(0.3249 - 0.1584)} = 3.939 \quad \text{and} \quad \frac{(0.7265)^2 - (0.1584)(0.3249)}{0.7265(0.3249 - 0.1584)} = 3.937$$

which is 3.937. We now have a prototype lowpass filter in Fig. 12.16b with $\hat{G}_p = -2.1$ dB, $\hat{G}_s = -20$ dB, $\omega_p' = 1$, and $\omega_s' = 3.937$.

Step 2.2: Determine the filter order n

The order of the Butterworth filter from Eq. (7.39) is

$$n = \frac{1}{2 \log 3.937} \log \left[\frac{10^2 - 1}{10^{0.21} - 1} \right] = 1.8498$$

which is rounded up to $n = 2$.

Step 2.3: Determine the cutoff frequency ω_c' of the prototype filter

In this step (which is not necessary for the Chebyshev design), we determine the 3-dB cutoff frequency ω_c' for the prototype filter using any one of the Eqs. (7.40) or (7.41).

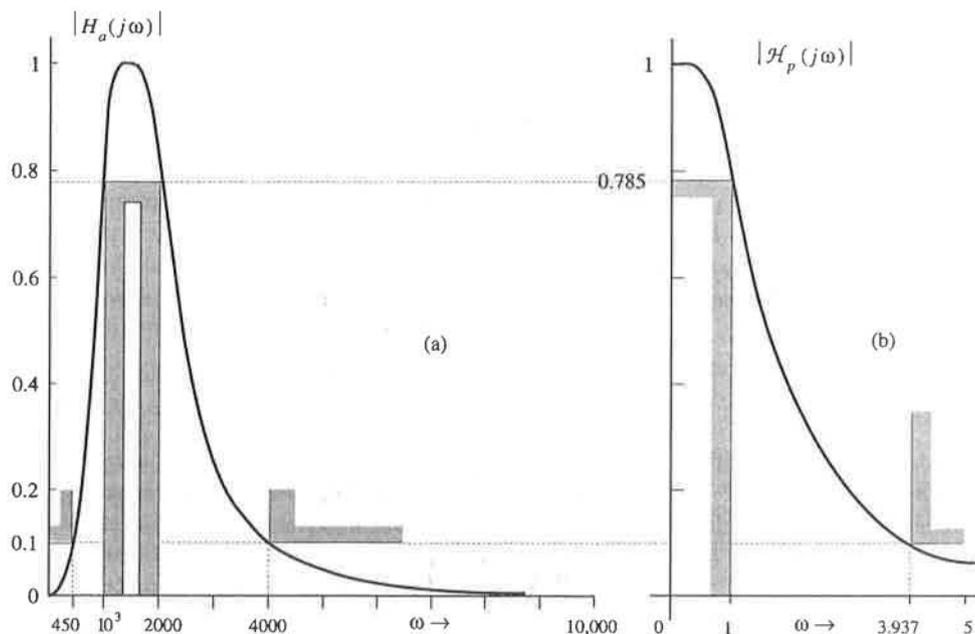


Fig. 12.16 Butterworth bandpass filter design using the bilinear transformation with prewarping.

Each equation gives a different answer, in general. However either answer will satisfy the specifications. Let us select Eq. (7.41), which yields

$$\omega_c' = \frac{\omega_s'}{(10^2 - 1)^{1/4}} = 1.248$$

Step 2.4: Find the normalized filter transfer function

The normalized second-order Butterworth filter transfer function (from Table 7.1) is

$$\mathcal{H}(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Step 2.5: Find the prototype filter transfer function $\mathcal{H}_p(s)$

The prototype filter transfer function $\mathcal{H}_p(s)$ is obtained by substituting s with $s/\omega_c' = s/1.248$ in the normalized transfer function $\mathcal{H}(s)$ found in step 4. This substitution yields

$$\mathcal{H}_p(s) = \frac{(1.248)^2}{s^2 + \sqrt{2}(1.248)s + (1.248)^2} = \frac{1.5575}{s^2 + 1.7649s + 1.5575}$$

Step 3: Find $H[z]$ by using the bilinear transformation

Finally, the desired transfer function $H[z]$ of the bandpass filter is obtained from $\mathcal{H}_p(s)$ by replacing s with $T_{bp}[z]$ from Eqs. (12.67) and (12.68). From Eq. (12.68), we obtain

$$a = \frac{\omega_{p_1}'\omega_{p_2}' - 1}{\omega_{p_1}'\omega_{p_2}' + 1} = \frac{-0.9485}{1.0515} = -0.9021 \quad \text{and} \quad b = \frac{\omega_{p_2}' - \omega_{p_1}'}{\omega_{p_1}'\omega_{p_2}' + 1} = \frac{0.1665}{1.0515} = 0.1583$$

Substitution of these values in Eq. (12.67) yields

$$T_{bp}[z] = \frac{z^2 + 2az + 1}{b(z^2 - 1)} = \frac{6.317(z^2 - 1.8042z + 1)}{z^2 - 1}$$

The desired bandpass filter transfer function $H[z]$ is obtained from $\mathcal{H}_p(s)$ by substituting s with $T_{bp}[z]$:

$$H[z] = \frac{0.02964(z^2 - 1)^2}{z^4 - 3.119z^3 + 3.926z^2 - 2.354z + 0.576}$$

The amplitude response $|H[e^{j\omega T}]|$ of this filter is illustrated in Fig. 12.16a. ■

⊙ Computer Example C12.7

Design a bandpass digital filter to meet the specifications in Example 12.8, using bilinear transformation with prewarping.

As before, we shall give here MATLAB functions to design the four basic types of approximations. In this case, W_p and W_s are 2-element vectors: $W_p = [1000 \ 2000]$ and $W_s = [450 \ 4000]$. The input data asks for frequencies so normalized that the sampling radian frequency is 2. As explained in Example C12.5, all the radian frequencies can be normalized by multiplying each of them by T/π . In the present case, $T = \pi/10^4$ so that $T/\pi = 1/10^4$. Thus, to normalize any radian frequency, we just divide it by 10^4 . The normalized W_p and W_s are $[0.1 \ 0.2]$ and $[0.045 \ 0.4]$, respectively.

```
Wp=[0.1 0.2];Ws=[0.045 0.4];Gp=-2.1;Gs=-20;
```

```
% Butterworth
```

```
[n,Wn]=buttord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=butter(n,Wn)
```

```
% Chebyshev
```

```
[n,Wn]=cheb1ord(Wp,Ws,-Gp,-Gs)
```

```
[b,a]=cheby1(n,-Gp,Wn)
```

```
% Inverse Chebyshev
```

```
[n,Wn]=cheb2ord(Wp,Ws,-Gp,-Gs)
```

```
[b,a]=cheby2(n,-Gs,Wn)
```

```
% Elliptic
```

```
[n,Wn]=ellipord(Wp,Ws,-Gp,-Gs)
```

```
[b,a]=ellip(n,-Gp,-Gs,Wn)
```

MATLAB gives $b=0.0296 \ 0 \ -0.0593 \ 0 \ 0.0296$ and $a=1 \ -3.119 \ 3.9259 \ -2.3539 \ 0.576$ for Butterworth option. Therefore

$$H[z] = \frac{0.0296(z^2 - 1)^2}{z^4 - 3.119z^3 + 3.9259z^2 - 2.3539z + 0.5760}$$

a result, which agrees with the answer found in Example 12.8. To plot the amplitude and the phase response, we can use the last 9 functions in Example C12.5. ⊙

⊙ Computer Example C12.8

Using bilinear transformation with prewarping, design a bandstop digital filter to meet the following specifications: $W_p = [450 \ 4000]$, $W_s = [1000 \ 2000]$, $\hat{G}_p = -2.1$ dB, and $\hat{G}_s = -20$ dB. Use $T = \pi/10^4$.

As before, we give here MATLAB functions to design the four types of approximations. The input data asks for frequencies so normalized that the sampling radian frequency is 2. As explained in Example C12.5, all the radian frequencies can be normalized by multiplying each of them by T/π . In the present case, $T = \pi/10^4$ so that $T/\pi = 1/10^4$. Thus, to normalize any radian frequency, we just divide it by 10^4 .

```
Ws=[0.1 0.2];Wp=[0.045 0.4];Gp=-2.1;Gs=-20;
```

```
% Butterworth
```

```
[n,Wn]=buttord(Wp,Ws,-Gp,-Gs)
```

```
[b,a]=butter(n,Wn,'stop')
```

```
% Chebyshev
```

```
[n,Wn]=cheb1ord(Wp,Ws,-Gp,-Gs)
```

```
[b,a]=cheby1(n,-Gp,Wn,'stop')
```

```
% Inverse Chebyshev
```

```
[n,Wn]=cheb2ord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=cheby2(n,-Gs,Wn,'stop')
```

```
% Elliptic
```

```
[n,Wn]=ellipord(Wp,Ws,-Gp,-Gs);
```

```
[b,a]=ellip(n,-Gp,-Gs,Wn,'stop')
```

MATLAB returns $b=0.3762 \ -1.3575 \ 1.9711 \ -1.3575 \ 0.3762$ and $a=1 \ -2.2523 \ 2.0563 \ -1.2053 \ 0.4197$ for Chebyshev option. Therefore

$$H[z] = \frac{0.3762(z^4 - 3.6084z^3 + 5.2395z^2 - 3.6084z + 1)}{z^4 - 2.2523z^3 + 2.0563z^2 - 1.2053z + 0.4197}$$

To plot the amplitude and the phase response, we can use the last 9 functions in Example C12.5. \odot

12.7 Nonrecursive Filters

The recursive filters are very sensitive to coefficient accuracy. Inaccuracies in their implementation, especially too short a word length, may change their behavior drastically and even make them unstable. Moreover, the recursive filter designs are well established only for amplitude responses that are piecewise constant, such as lowpass, bandpass, highpass, and bandstop filters. In contrast, a nonrecursive filter can be designed to have an arbitrarily shaped frequency response. In addition, nonrecursive filters can be designed to have a linear phase response. On the other hand, if a recursive filter can be found to do the job of a nonrecursive filter, the recursive filter is of lower order; that is, it is faster (with less processing delay) and requires less memory. If processing delay is not critical, the nonrecursive filter is the obvious choice. They also have an important place in non-audio applications, where a linear phase response is important. We shall review the concept of nonrecursive systems briefly.

As discussed in Sec. 12.3, nonrecursive filters may be viewed as recursive filters, where all the feedback or recursive coefficients are zero; that is, when

$$a_0 = a_1 = a_2 = \cdots = a_{n-1} = 0$$

Consequently, the transfer function of the resulting n th-order nonrecursive filter is

$$H[z] = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0}{z^n} \quad (12.69a)$$

$$= b_n + \frac{b_{n-1}}{z} + \frac{b_{n-2}}{z^2} + \cdots + \frac{b_1}{z^{n-1}} + \frac{b_0}{z^n} \quad (12.69b)$$

Now, by definition, $H[z]$ is the z -transform of $h[k]$:

$$\begin{aligned} H[z] &= \sum_{k=0}^{\infty} h[k] z^{-k} \\ &= h[0] + \frac{h[1]}{z} + \frac{h[2]}{z^2} + \cdots + \frac{h[n]}{z^n} + \frac{h[n+1]}{z^{n+1}} + \cdots \end{aligned} \quad (12.70)$$

Comparison of this equation with Eq. (12.69b) shows that $h[k] = 0$ for $k > n$, and Eq. (12.70) becomes

$$H[z] = h[0] + \frac{h[1]}{z} + \frac{h[2]}{z^2} + \cdots + \frac{h[n]}{z^n} \quad (12.71a)$$

$$= \frac{h[0]z^n + h[1]z^{n-1} + \cdots + h[n-1]z + h[n]}{z^n} \quad (12.71b)$$

where

$$h[k] = \begin{cases} b_{n-k} & 0 \leq k \leq n \\ 0 & k > n \end{cases} \quad (12.72)$$

The impulse response $h[k]$ has a finite width of $(n+1)$ elements. Hence, these filters are finite impulse response (FIR) filters. We shall use the terms nonrecursive and FIR interchangeably. Similarly, the terms recursive and IIR (infinite impulse response) will be used interchangeably in our future discussion.

The impulse response $h[k]$ can be expressed as

$$h[k] = h[0]\delta[k] + h[1]\delta[k-1] + \cdots + h[n]\delta[k-n] \quad (12.73)$$

The frequency response of this filter is obtained from Eq. (12.71a) as

$$H[e^{j\omega T}] = h[0] + h[1]e^{-j\omega T} + \cdots + h[n]e^{-jn\omega T} \quad (12.74a)$$

$$= \sum_{k=0}^n h[k]e^{-jk\omega T} \quad (12.74b)$$

Filter Realization

The nonrecursive (FIR) filter in Eq. (12.69a) is a special case of a general filter with all feedback (or recursive) coefficients zero. Therefore, the realization of this filter is the same as that of the n th-order recursive filter with all the feedback connections omitted. Figure 12.7b shows a canonical realization of this filter. It is easy to verify from this figure that for the input $\delta[k]$, the output is $h[k]$ given in Eq. (12.73).

The filter in Fig. 12.7b is a tapped delay line with successive taps at unit delay (T seconds). Such a filter is known as a **transversal filter**. Tapped analog delays are integrated circuits, which are available commercially. In these circuits the time delay is implemented by using charge transfer devices, which sample the input signal every T seconds (unit delay) and transfer the successive values of the samples to m storage cells. The stored signal at the k th tap is the input signal delayed by k time units (kT seconds). The sampling interval can be varied electronically over a wide range. Time delay can also be obtained by using shift registers.

12.7-1 Symmetry Conditions for Linear Phase Response

Consider an n th-order finite impulse response (FIR) filter described by the transfer function $H[z]$ [Eq. (12.69) or (12.71)] and the corresponding impulse response $h[k]$ [Eq. (12.73)]. We now show that if $h[k]$ is either symmetric (Fig. 12.17a) or antisymmetric (Fig. 12.17b) about its center point, the filter phase response is a linear function of ω . We consider a case where n is even. To avoid too much abstractness, we choose some convenient value for n , say $n = 4$, to demonstrate our point. It will then be easier to understand the generalization to the n th-order case.

For $n = 4$, the impulse response in Eq. (12.73) reduces to

$$h[k] = h[0]\delta[k] + h[1]\delta[k-1] + h[2]\delta[k-2] + h[3]\delta[k-3] + h[4]\delta[k-4]$$

The transfer function $H[z]$ in Eq. (12.71b) reduces to

$$H[z] = h[0] + \frac{h[1]}{z} + \frac{h[2]}{z^2} + \frac{h[3]}{z^3} + \frac{h[4]}{z^4} \quad (12.75a)$$

$$= z^{-2} \left(h[0]z^2 + h[1]z + h[2] + h[3]z^{-1} + h[4]z^{-2} \right) \quad (12.75b)$$

Therefore, the frequency response is

$$H[e^{j\omega T}] = e^{-j2\omega T} \left(h[0]e^{j2\omega T} + h[1]e^{j\omega T} + h[2] + h[3]e^{-j\omega T} + h[4]e^{-j2\omega T} \right) \quad (12.76)$$

If $h[k]$ is symmetric about its center point ($k = 2$ in this case), then

$$h[0] = h[4], \quad h[1] = h[3]$$

and the frequency response reduces to

$$\begin{aligned} H[e^{j\omega T}] &= e^{-j2\omega T} \left(h[0] [e^{j2\omega T} + e^{-j2\omega T}] + h[2] + h[1] [e^{j\omega T} + e^{-j\omega T}] \right) \\ &= e^{-j2\omega T} \left(h[2] + 2h[1] \cos \omega T + 2h[0] \cos 2\omega T \right) \end{aligned} \quad (12.77)$$

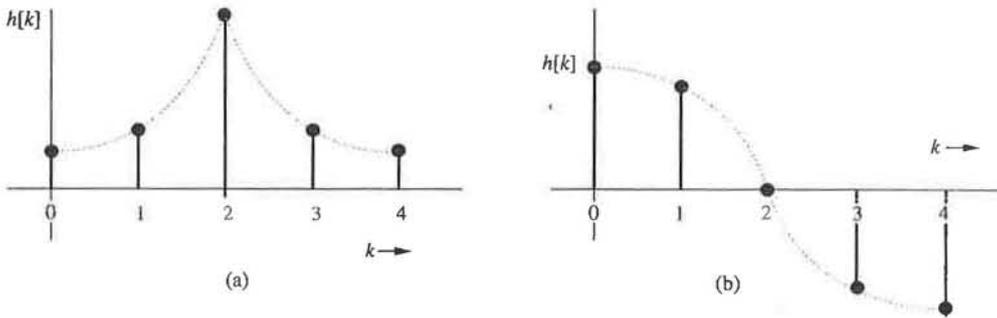


Fig. 12.17 Symmetry conditions for linear phase frequency response in nonrecursive filters.

The quantity inside the parenthesis is real; it may be positive over some bands of frequencies and negative over other bands. This quantity represents the amplitude response $|H[e^{j\omega T}]|$.† The phase response is given by

$$\angle H[e^{j\omega T}] = -2\omega T$$

The phase response is a linear function of ω . The time delay is the negative of the slope of $\angle H[e^{j\omega T}]$ with respect to ω , which is $2T$ seconds in this case [see Eq. (4.59)].

If $h[k]$ is antisymmetric about its center point, then the antisymmetry about the center point requires that $h[k] = 0$ at the center point‡ [see Fig. 12.17b]. Thus, in this case

$$h[0] = -h[4], \quad h[1] = -h[3], \quad h[2] = 0$$

and the frequency response reduces to

$$\begin{aligned} H[e^{j\omega T}] &= e^{-j2\omega T} \left(h[0](e^{j2\omega T} - e^{-j2\omega T}) + h[1](e^{j\omega T} - e^{-j\omega T}) \right) \\ &= 2je^{-j2\omega T} \left(h[1] \sin \omega T + h[0] \sin 2\omega T \right) \\ &= 2e^{j(\frac{\pi}{2} - 2\omega T)} \left(h[1] \sin \omega T + h[0] \sin 2\omega T \right) \end{aligned}$$

Thus, the phase response in this case is

$$\angle H[e^{j\omega T}] = \frac{\pi}{2} - 2\omega T$$

†Strictly speaking, $|H[e^{j\omega T}]|$ cannot be negative. Recall, however, that the only restriction on amplitude is that it cannot be complex. It has to be real; it can be positive or negative. We should have used some other notation such as $A(\omega)$ to denote the amplitude response. But this would create too many related functions causing possible confusion. Another alternative is to incorporate the negative sign of the amplitude in the phase response, which will be increased (or decreased) by π over the band where the amplitude response is negative. This alternative will still maintain the phase linearity.

‡Antisymmetry property requires that $h[k] = -h[-k]$ at the center point also. This condition is possible only if $h[k] = 0$ at this point.

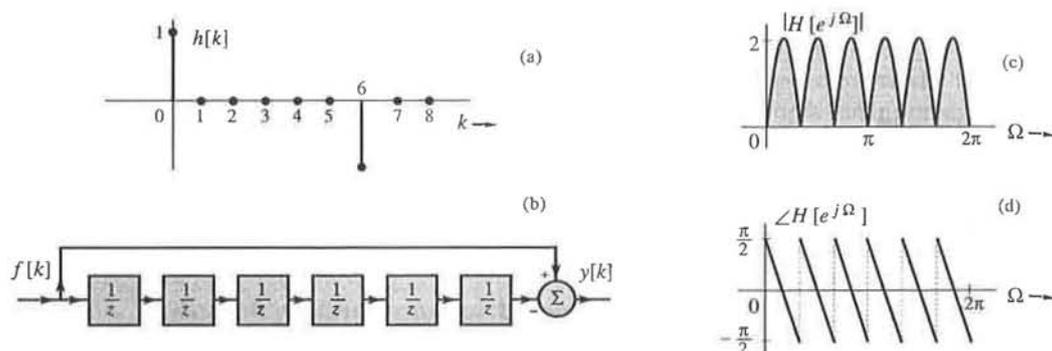


Fig. 12.18 Comb filter: Its impulse and frequency response.

The phase response in this case is also a linear function of ω . The system has the time delay (the negative slope of $\angle H[e^{j\omega T}]$ with respect to ω) of $2T$ seconds (2 units), the same as in the symmetric case. The only difference is that the phase response has a constant term $\pi/2$.

We can obtain similar results for odd values of n (see Prob. 12.7-1). This result can be generalized for an n th-order case to show that the phase response is linear, and the time delay is $\frac{nT}{2}$ seconds (or $\frac{n}{2}$ units).

Example 12.9: Comb Filter

Determine the transfer function and the frequency response of a sixth-order comb filter whose impulse response is given by

$$h[k] = \delta[k] - \delta[k - 6]$$

This impulse response is illustrated in Fig. 12.18a. Its canonical realization is depicted in Fig. 12.18b. Observe that $h[k]$ is antisymmetric about $k = 3$. Also

$$H[z] = \sum_{-\infty}^{\infty} f[k]z^{-k} = 1 - z^{-6} = \frac{z^6 - 1}{z^6} \quad (12.78)$$

The frequency response is given by

$$\begin{aligned} H[e^{j\Omega}] &= 1 - e^{-j6\Omega} \\ &= e^{-j3\Omega}(e^{j3\Omega} - e^{-j3\Omega}) \\ &= 2je^{-j3\Omega} \sin 3\Omega \\ &= 2e^{j(\frac{\pi}{2} - 3\Omega)} \sin 3\Omega \end{aligned}$$

The term $\sin 3\Omega$ can be positive as well as negative. Therefore

$$|H[e^{j\Omega}]| = 2|\sin 3\Omega|$$

and $\angle H[e^{j\Omega}]$ is as indicated in Fig. 12.8d. The amplitude response, illustrated in Fig. 12.18c, is shaped like a comb with periodic nulls. The filter can be realized by the structure in Fig. 12.7b. Since $h[k] = b_{6-k}$ [see Eq. (12.72)], $b_0 = -1$, $b_1 = b_2 = b_3 = b_4 = b_5 = 0$, and $b_6 = 1$.

Using the same argument, the reader can show that an n th-order comb filter transfer function is

$$H[z] = \frac{z^n - 1}{z^n} \quad (12.79)$$

and

$$H[e^{j\Omega}] = 2e^{j(\frac{\pi}{2} - \frac{n\Omega}{2})} \sin \frac{n\Omega}{2} \quad \blacksquare$$

12.8 Nonrecursive Filter Design

As in the case of recursive (IIR) filters, nonrecursive filters can be designed by using the time-domain and the frequency-domain equivalence criteria. In the time-domain equivalence criterion, the digital filter impulse response is made identical to the samples of the desired (analog) filter impulse response. In the frequency-domain equivalence criterion, the digital filter frequency response samples at uniform frequency intervals are matched to the desired analog filter frequency response samples. This method is also known as the **frequency sampling** or the **spectral sampling** method.

12.8-1 Time-Domain Equivalence Method of FIR Filter Design

The time-domain equivalence method (also known as the **Fourier series** method) of design of FIR filters is identical to that for IIR filters discussed in Sec. 12.5, except that FIR filter impulse response must be of finite duration. Therefore, the desired impulse response must be truncated to have finite duration. Truncating the impulse response abruptly will result in oscillatory frequency response because of the Gibbs phenomenon discussed in Sec. 3.4-3. In some filtering applications the oscillatory frequency response (which decays slowly as $1/\omega$) in the stopband may not be acceptable. By using a tapered window function for truncation of $h[k]$, the oscillatory behavior can be reduced or even eliminated at the cost of increasing the transition band as discussed in Sec. 4.9. Note that the impulse response of an n th-order FIR filter has $n + 1$ samples. Hence, for truncating $h[k]$ for an n -th order filter, we must use an N_0 -point window, where $N_0 = n + 1$. Several window functions and their tradeoffs appear in Table 12.2.

Design Procedure

Much of the discussion so far has been rather general. We shall now give some concrete examples of such filter design. Because we want the reader to be focussed on the procedure, we shall intentionally choose a small value for n (the filter order) to avoid getting distracted by a jungle of data. The procedure, however, is general and it can be applied to any value of n .

The steps in the time-domain equivalence design method are:

1. Determine the filter impulse response $h[k]$

In the first step, we find the impulse response $h[k]$ of the desired filter. According to the time-domain equivalence criterion in Eq. (12.31),

$$h[k] = Th_a(kT) \quad (12.80)$$

TABLE 12.2
Some Window Function and Their Characteristics

Window $w[k]$	Mainlobe Width	Rolloff Rate dB/octave	Peak Sidelobe Level in dB
$-M \leq k \leq M \quad M = \frac{N_0-1}{2}$ 1 Rectangular: $\text{rect}\left(\frac{k}{N_0-1}\right)$	$\frac{4\pi}{N_0}$	-6	-13.3
2 Bartlett: $\Delta\left(\frac{k}{N_0-1}\right)$	$\frac{8\pi}{N_0}$	-12	-26.5
3 Hanning: $0.5\left[1 + \cos\left(\frac{2\pi k}{N_0-1}\right)\right]$	$\frac{8\pi}{N_0}$	-18	-31.5
4 Hamming: $0.54 + 0.46 \cos\left(\frac{2\pi k}{N_0-1}\right)$	$\frac{8\pi}{N_0}$	-6	-42.7
5 Blackman: $0.42 + 0.5 \cos\left(\frac{2\pi k}{N_0-1}\right) + 0.08 \cos\left(\frac{4\pi k}{N_0-1}\right)$	$\frac{12\pi}{N_0}$	-18	-58.1
6 Kaiser: $\frac{I_0\left[\alpha\sqrt{1-4\left(\frac{k}{N_0-1}\right)^2}\right]}{I_0(\alpha)} \quad 1 \leq \alpha \leq 10$	$\frac{11.2\pi}{N_0}$	-6	-59.9 ($\alpha = 8.168$)

where $h_a(t)$ is the impulse response of the analog filter $H_a(s)$. Therefore, $h_a(t)$ is the inverse Laplace transform of $H_a(s)$ or the inverse Fourier transform of $H_a(j\omega)$. Thus,

$$h_a(t) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} H_a(j\omega) e^{j\omega t} d\omega \quad (12.81a)$$

Recall that a digital filter frequency response is periodic with the first period in the frequency range $-\frac{\pi}{T} \leq \omega < \frac{\pi}{T}$. Hence, the best we could hope is to realize the equivalence of $H_a(j\omega)$ within this range. For this reason, the limits of integration are taken from $-\pi/T$ to π/T . Therefore, according to Eq. (12.80)

$$h[k] = T h_a(kT) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} H_a(j\omega) e^{j\omega kT} d\omega \quad (12.81b)$$

2. Windowing

For linear phase filters, we generally start with zero phase filters for which $H_a(j\omega)$ is either real or imaginary. The impulse response $h_a(t)$ is either an even or odd function of t (see Prob. 4.1-1). In either case, $h_a(t)$ is centered at $t = 0$ and has infinite duration in general. But $h[k]$ must have only a finite duration and it must start at $k = 0$ (causal) for filter realizability. Consequently, the $h[k]$ found in step 1 needs to be truncated using an N_0 -point window and then

delayed by $\frac{N_0-1}{2}$ to make it causal. This delay produces the desired linear-phase frequency response.

Straight truncation of data amounts to using a rectangular window, which has a unit weight over the window width, and zero weight for all other k . We saw that although such a window gives the smallest transition band, it results in a slowly decaying oscillatory frequency response in the stopband. This behavior can be corrected by using a tapered window of a suitable width.

*

3. Filter Frequency Response and Realization

Knowing $h[0], h[1], h[2], \dots, h[n]$, we determine $H[z]$ using Eq. (12.71) and the frequency response $H[e^{j\omega T}]$ using Eq. (12.74). We now realize the truncated $h[k]$ using the structure in Fig. 12.7b.

Optimality of the Procedure

The procedure outlined here using a rectangular window function is the optimum in the sense that the energy of the error (difference) between the desired frequency response $H_a(j\omega)$ and the realized frequency response $H[e^{j\omega T}]$ is the minimum for a given N_0 . This conclusion follows from the fact that the resulting filter frequency response $H[e^{j\omega T}]$ is given by

$$H[e^{j\omega T}] = \sum_k h[k]e^{-j\omega kT}$$

This frequency response is an approximation of the desired frequency response $H_a(j\omega)$ because of the truncation of $h[k]$. Thus,

$$H_a(j\omega) \approx \sum_k h[k]e^{-j\omega kT}$$

How do we select $h[k]$ for the best approximation in the sense of minimizing the energy of the error $H_a(j\omega) - H[e^{j\omega T}]$? We have already solved this problem in Sec. 3.3-2. The above equation shows that the right-hand side is the finite term exponential Fourier series for $H_a(j\omega)$ with period $2\pi/T$. As seen from Eq. (12.81b), $h[k]$ are the Fourier coefficients. We also know that a finite Fourier series is the optimum (in the sense of minimizing the error energy) for a given N_0 according to the finality property of the Fourier coefficients discussed in Sec. 3.3-2.† Clearly, this choice of $h[k]$ is optimum in the sense of the minimum mean squared error. For the obvious reason, this method is also known as the Fourier series method.

■ Example 12.10

Design an ideal lowpass filter for audio band with cutoff frequency 20 kHz. Use a sixth-order nonrecursive filter using rectangular and Hamming windows. The highest frequency to be processed is $\mathcal{F}_h = 40$ kHz.

In this case $n = 6$ and $N_0 = n + 1 = 7$. First we shall choose a suitable value for T . According to Eq. (12.58)

$$T \leq \frac{\pi}{\omega_h} = \frac{1}{2\mathcal{F}_h} = 12.5 \times 10^{-6}$$

†Note that this finite term Fourier series corresponds to the rectangular window function. For windows other than rectangular, the optimality does not hold.

Recall that a continuous-time sinusoid of frequency ω , during digital processing appears as a discrete-time sinusoid of frequency $\Omega = \omega T$. The cutoff frequency $\omega_c = 2\pi(20,000) = 40,000\pi$ appears as a discrete-time sinusoid of frequency

$$\Omega_c = \omega_c T = 40,000\pi(12.5 \times 10^{-6}) = \frac{\pi}{2}, \text{ and } \omega_c = \frac{\pi}{2T}$$

The desired (zero phase) filter frequency response is depicted in Fig. 12.19a on both ω and Ω scales. Recall that the digital frequency range is from $-\pi$ to π only. We wish to design an ideal lowpass filter of cutoff frequency $\omega_c = \frac{\pi}{2T}$ rad/s. The frequency response has a period of 2π on Ω scale, and $2\pi/T$ on ω scale. Rather than substitute $T = 12.5 \times 10^{-6}$, it is convenient to leave T as an unknown in our computations and substitute the value only in the end. Thus, we shall use the radian cutoff frequency $\omega_c = \pi/2T$.

The impulse response of the desired ideal lowpass (zero phase) filter in Fig. 12.19a is (Table 4.1, Pair 18)

$$h_a(t) = \frac{1}{2T} \operatorname{sinc}\left(\frac{\pi t}{2T}\right) \quad (12.82)$$

and according to the impulse invariance criterion [Eq. (12.39)]

$$h[k] = Th_a(kT) = \frac{1}{2} \operatorname{sinc}\left(\frac{\pi kT}{2T}\right) = \frac{1}{2} \operatorname{sinc}\left(\frac{\pi k}{2}\right) \quad (12.83)$$

Figure 12.19b shows $h[k]$. To make this filter realizable, we need to truncate it using a suitable N_0 -point window, then delay the truncated $h[k]$ by $\frac{N_0-1}{2}$ units. In the present example, $N_0 = 7$. Figure 12.19c shows the impulse response truncated by a 7-point rectangular window and Fig. 12.19d shows the truncated $h[k]$ delayed by $\frac{N_0-1}{2} = 3$ units.

Note that the noncausal filter in this case is made realizable at the cost of a delay of $t = 3T$ seconds. This constant delay of $\frac{N_0-1}{2}$ is what produces a linear phase characteristic. The rectangular windowed, causal filter impulse response $h_R[k]$ is the truncated $h[k]$ in Fig. 12.19d delayed by $3T$.

$$h_R[k] = h[k-3] = \frac{1}{2} \operatorname{sinc}\left[\frac{\pi(k-3)}{2}\right] \quad 0 \leq k \leq 6 \quad (12.84)$$

The values of the coefficient $h_R[k]$ are shown in Table 12.3. Also

$$\begin{aligned} H[z] &= \sum_{k=0}^6 h[k]z^{-k} \\ &= -\frac{1}{3\pi} + \frac{1}{\pi}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{\pi}z^{-4} - \frac{1}{3\pi}z^{-6} \\ &= z^{-3} \left(-\frac{1}{3\pi}z^3 + \frac{1}{\pi}z + \frac{1}{2} + \frac{1}{\pi}z^{-1} - \frac{1}{3\pi}z^{-3} \right) \end{aligned} \quad (12.85)$$

Hence, the frequency response $H_R[e^{j\omega T}]$ is

$$\begin{aligned} H_R[e^{j\omega T}] &= \sum_{k=0}^6 h_R[k]e^{-j\omega kT} \\ &= e^{-j3\omega T} \left[\frac{1}{2} + \frac{1}{\pi} (e^{j\omega T} + e^{-j\omega T}) - \frac{1}{3\pi} (e^{j3\omega T} + e^{-j3\omega T}) \right] \\ &= e^{-j3\omega T} \left[\frac{1}{2} + \frac{2}{\pi} \cos \omega T - \frac{2}{3\pi} \cos 3\omega T \right] \\ &= e^{-j3\omega T} \left[\frac{1}{2} + \frac{2}{\pi} \cos \left(\frac{\omega}{80,000} \right) - \frac{2}{3\pi} \cos \left(\frac{3\omega}{80,000} \right) \right] \end{aligned} \quad (12.86)$$

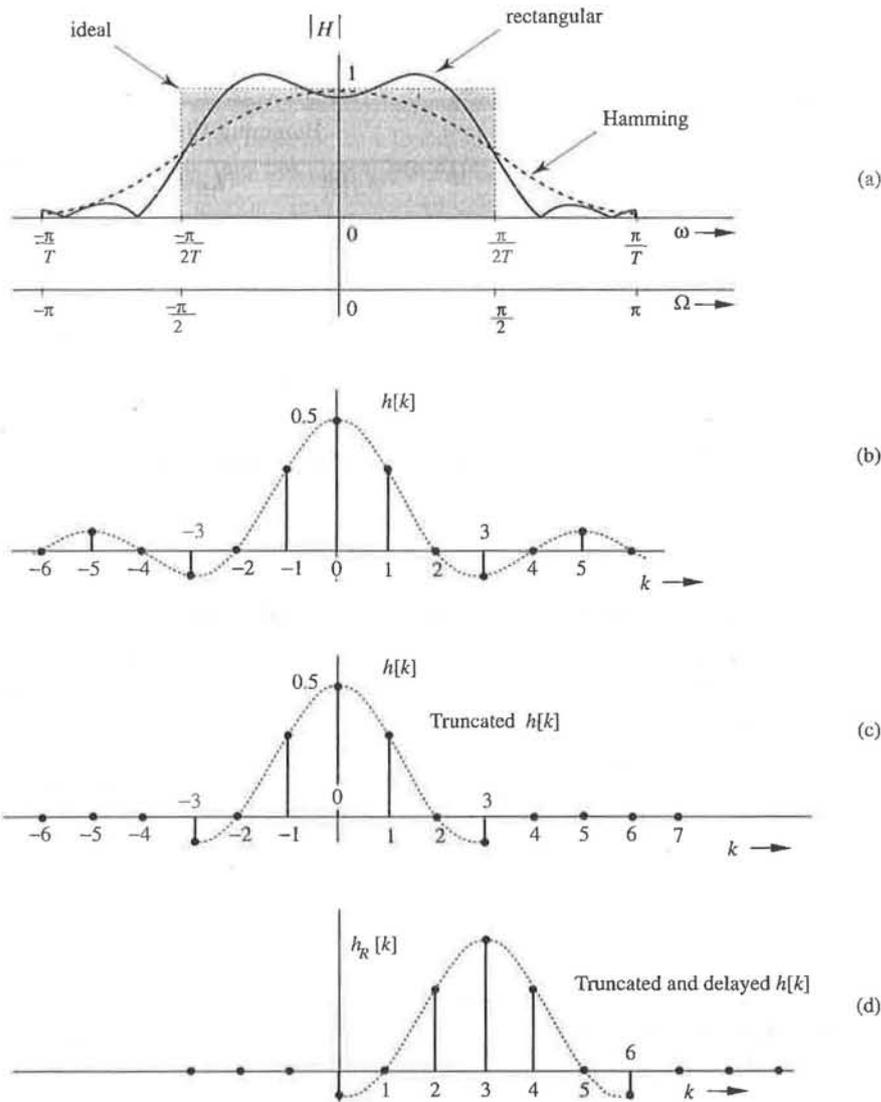


Fig. 12.19 Nonrecursive method of lowpass filter design.

The term $e^{-j3\omega T}$ is a linear phase representing the delay of $3T$ seconds. The magnitude of the bracketed term, depicted in Fig. 12.19a by a solid curve, exhibits oscillatory behavior which decays rather slowly over the stopband. Although increasing n (the system order) improves the frequency response, its oscillatory nature persists (Gibbs phenomenon).[†] In some filtering applications, the oscillatory characteristic (which decays slowly as $1/\omega$) in the stopband may not be acceptable. By using a tapered window func-

[†]Eq. (12.86) is identical to the first three terms in Eq. (3.61) except that the former is a function of ω and the latter is a function of t . Clearly, $H[e^{j\omega T}]$ is a truncated Fourier series for the gate function. As we increase n , $H[e^{j\omega T}]$ converges to the gate function in the manner depicted in Fig. 3.11. Regardless of the value of n , however, $H[e^{j\omega T}]$ exhibits oscillatory behavior because of the Gibbs phenomenon.

TABLE 12.3

Rectangular Window		Hamming Window	
k	$h_R[k]$	$w_H[k]$	$h_H[k]$
0	$-1/3\pi$	0.08	-0.00848
1	0	0.31	0
2	$1/\pi$	0.77	0.245
3	$1/2$	1	0.5
4	$1/\pi$	0.77	0.245
5	0	0.31	0
6	$-1/3\pi$	0.08	-0.00848

tion such as a Hamming window, the oscillatory behavior can be eliminated at the cost of increasing the transition band (from passband to stopband). The Hamming window function is given by

$$w_H[k] = \begin{cases} 0.54 + 0.46 \cos\left(\frac{2\pi k}{N_0-1}\right) & -\frac{N_0-1}{2} \leq |k| \leq \frac{N_0-1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (12.87)$$

In our case $N_0 = n + 1 = 7$. Hence,

$$w_H[k] = \begin{cases} 0.54 + 0.46 \cos\left(\frac{\pi k}{3}\right) & -3 \leq |k| \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Table 12.3 also shows the (delayed) Hamming window coefficients $w_H[k]$ and the corresponding impulse response $h_H[k] = h[k]w_H[k]$. The frequency response of the Hamming window filter is

$$\begin{aligned} H_H[e^{j\omega T}] &= e^{-j3\omega T} \left[\frac{1}{2} + 0.245 (e^{j\omega T} + e^{-j\omega T}) - 0.00848 (e^{j3\omega T} + e^{-j3\omega T}) \right] \\ &= e^{-j3\omega T} \left[\frac{1}{2} + 0.49 \cos \omega T - 0.01696 \cos 3\omega T \right] \end{aligned}$$

With the coefficients $h_R[k]$ (or $h_H[k]$) in Table 12.3, the desired filter can be realized by using six delay elements, as depicted in Fig. 12.7.

According to Eqs. (12.86), we have

$$|H_R[e^{j\omega T}]| = \left| \frac{1}{2} + \frac{2}{\pi} \cos \omega T - \frac{2}{3\pi} \cos 3\omega T \right| \quad (12.88a)$$

and

$$\angle H_R[e^{j\omega T}] = \begin{cases} -3\omega T & \text{when } \frac{1}{2} + \frac{2}{\pi} \cos \omega T - \frac{2}{3\pi} \cos 3\omega T \geq 0 \\ \pi - 3\omega T & \text{when } \frac{1}{2} + \frac{2}{\pi} \cos \omega T - \frac{2}{3\pi} \cos 3\omega T < 0 \end{cases} \quad (12.88b)$$

For the Hamming filter

$$|H_H[e^{j\omega T}]| = \frac{1}{2} + 0.49 \cos \omega T - 0.01696 \cos 3\omega T \quad \angle H_H[e^{j\omega T}] = -3\omega T \quad (12.88c)$$

In either filter, the phase response is a linear function of ω with slope $-3T$, indicating time delay of $3T$. Note that both $h_R[k]$ and $h_h[k]$ are symmetric about $k = 3$. ■

⊙ **Computer Example C12.9**

Using MATLAB, find the frequency response of the lowpass filter in Example 12.10 for 98th-order filter. Plot the frequency response for rectangular and Hamming window filters.

```

N0=99;
m=(N0-1)/2;
k=0:N0-1;
h1=(1/2)*sinc((k-m)/2);
num1=h1;
den1=[1, zeros(1,N0-1)];
W=-pi:pi/100:pi;
H1=freqz(num1,den1,W);
mag1=abs(H1);
phase1=180/pi*unwrap(angle(H1));
for i=1:N0
k=i-1;
h2(i)=(1/2)*sinc((k-m)/2)*(0.54+0.46*cos(pi*(k-m)/m));
end
num2=h2;
den2=[1, zeros(1,N0-1)];
W=-pi:pi/100:pi;
H2=freqz(num2,den2,W);
mag2=abs(H2);
phase2=180/pi*unwrap(angle(H2));
subplot(2,1,1);
plot(W,mag1,W,mag2);grid;
subplot(2,1,2);
plot(W,phase1,W,phase2);grid

```

△ **Exercise E12.6**

If we were to use $n = 8$ filter in Example 12.12, show that the filter transfer function for the bartlett (triangular) window is

$$H[z] = \frac{-0.02653z^6 + 0.2387z^4 + 0.5z^3 + 0.2387z^2 + 0.02653}{z^6}$$

Observe that in this case the filter order is reduced by 2 because the two end-points have a zero value for the Bartlett window. ▽

■ **Example 12.11**

Design a differentiator using a tenth-order nonrecursive filter.

In this case $n = 10$ and $N_0 = 11$. The transfer function of an ideal differentiator is $H_a(s) = s$. Therefore,

$$H_a(j\omega) = j\omega \quad |\omega| \leq \frac{\pi}{T}$$

and according to Eq. (12.81b)†

$$\begin{aligned} h[k] &= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} j\omega e^{j\omega kT} d\omega \\ &= \frac{\cos k\pi}{kT} \end{aligned} \quad (12.89)$$

This is a noncausal filter. We truncate it using an 11-point window and then delay it by $\frac{N_0-1}{2} = 5$ units to make it realizable. The desired impulse response using a rectangular window is

$$h_R[k] = h[k-5] \quad 0 \leq k \leq 10$$

Table 12.4 shows this impulse response under the rectangular window. The frequency response is

$$H_R[e^{j\omega T}] = \sum_{k=0}^{10} h_R[k] e^{-jkT\omega}$$

Substitution of values of $h_R[k]$ from Table 12.4 in this equation yields

$$\begin{aligned} H_R[e^{j\omega T}] &= \frac{1}{T} e^{-j5\omega T} [(e^{j\omega T} - e^{-j\omega T}) - \frac{1}{2}(e^{j2\omega T} - e^{-j2\omega T}) + \frac{1}{3}(e^{j3\omega T} - e^{-j3\omega T}) \\ &\quad - \frac{1}{4}(e^{j4\omega T} - e^{-j4\omega T}) + \frac{1}{5}(e^{j5\omega T} - e^{-j5\omega T})] \\ &= \frac{j}{T} e^{-j5\omega T} [2 \sin \omega T - \sin 2\omega T + \frac{2}{3} \sin 3\omega T - \frac{1}{2} \sin 4\omega T + \frac{2}{5} \sin 5\omega T] \\ &= \frac{1}{T} e^{j(\frac{\pi}{2}-5\omega T)} [2 \sin \omega T - \sin 2\omega T + \frac{2}{3} \sin 3\omega T - \frac{1}{2} \sin 4\omega T + \frac{2}{5} \sin 5\omega T] \end{aligned}$$

Hence

$$|H_R[e^{j\omega T}]| = \frac{1}{T} |2 \sin \omega T - \sin 2\omega T + \frac{2}{3} \sin 3\omega T - \frac{1}{2} \sin 4\omega T + \frac{2}{5} \sin 5\omega T| \quad (12.90a)$$

and

$$\angle H_R[e^{j\omega T}] = \begin{cases} \frac{\pi}{2} - 5\omega T & \omega > 0 \\ -\frac{\pi}{2} + 5\omega T & \omega < 0 \end{cases} \quad (12.90b)$$

The terms $\pm 5\omega T$ represent the time delay of $5T$ seconds. Note that because $H_a(j\omega) = j\omega$,

$$\begin{aligned} |H_a(j\omega)| &= |\omega| \\ \angle H_a(j\omega) &= \begin{cases} \pi/2 & \omega > 0 \\ -\pi/2 & \omega < 0 \end{cases} \end{aligned} \quad (12.91)$$

†Note that according to Eq. (12.81a)

$$h_a(0) = \frac{1}{2\pi} \int_{-\pi/2T}^{\pi/2T} j\omega d\omega = 0$$

Therefore, $h[0] = 0$.

Therefore, the phase characteristic of $H_R[e^{j\omega T}]$ is identical to the desired (ideal) differentiator with a delay of $5T$ seconds. Figure 12.20 shows the amplitude response $|H|$ of the realized filter and the $|H_a|$ of the ideal (desired) differentiator. The amplitude response $|H_R[e^{j\omega T}]|$, which is oscillatory, can be improved by using a tapered window function to truncate $h_a(t)$ gradually. The Hamming window function $w_H[k]$ [Eq. (12.87)] for $n = 10$ is given by

$$w_H[k] = \begin{cases} 0.54 + 0.46 \cos\left(\frac{\pi k}{5}\right) & |k| \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

Table 12.4 shows the (delayed) Hamming coefficients $w_H[k]$ and the corresponding filter impulse response $h_H[k] = h[k]w_H[k]$. Note that $h_R[k]$ and $h_H[k]$ are both antisymmetric about $k = 5$. Using the Hamming coefficients from Table 12.4, $H_H[e^{j\omega T}]$, the windowed filter frequency response, is given by

$$H_H[e^{j\omega T}] = \frac{1}{T} e^{j(\frac{\pi}{2} - 5\omega T)} [1.8242 \sin \omega T - 0.682 \sin 2\omega T + 0.2652 \sin 3\omega T \\ - 0.0839 \sin 4\omega T + 0.032 \sin 5\omega T]$$

Figure 12.20 shows the amplitude response of the ideal, the rectangular-windowed, and the Hamming-windowed filter characteristics. The phase response of both the realized filters are identical, but with a delay of $5T$ seconds. The amplitude response of the windowed (Hamming) filter is practically the same over a large band (two thirds of the band) as that for the ideal filter, and hence is preferable to the unwinded (or rectangular-windowed) filter.

TABLE 12.4

Rectangular Window		Hamming Window	
k	$h_R[k]$	$w_H[k]$	$h_H[k]$
0	$1/5T$	0.08	$0.016/T$
1	$-1/4T$	0.1678	$-0.04195/T$
2	$1/3T$	0.3978	$0.1326/T$
3	$-1/2T$	0.6821	$-0.3410/T$
4	$1/T$	0.9121	$0.9121/T$
5	0	1	0
6	$-1/T$	0.9121	$-0.9121/T$
7	$1/2T$	0.6821	$0.3410/T$
8	$-1/3T$	0.3978	$-0.1326/T$
9	$1/4T$	0.1678	$0.04195/T$
10	$-1/5T$	0.08	$-0.016/T$

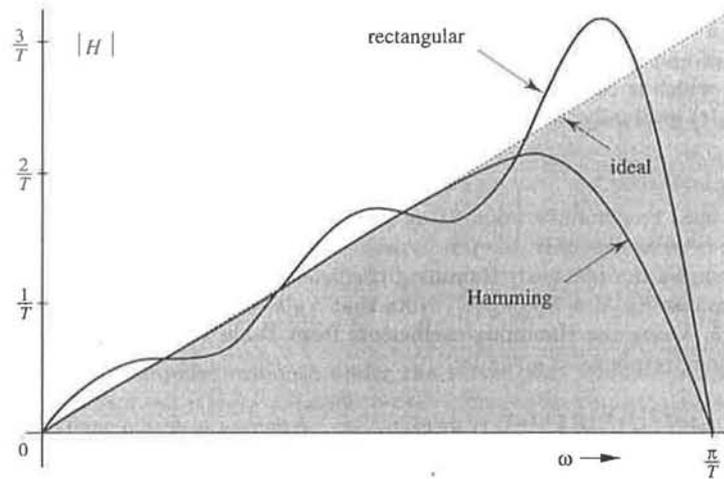


Fig. 12.20 An ideal differentiator design by a tenth-order nonrecursive filter using rectangular and Hamming windows.

The amplitude response of the Hamming filter is practically ideal up to frequency $\omega \approx (2/3)(\pi/T) = 2\pi/3T$. If we wish to design a digital differentiator for audio application, for instance, where the highest frequency is, say 20 kHz, we should select this frequency to be less than $2\pi/3T$. Thus,

$$2\pi \times 20,000 \leq \frac{2\pi}{3T} \implies T \leq 16.67 \mu\text{s}$$

Thus, a choice of $T \leq 16.67 \mu\text{s}$ would result in a desired differentiator. This example (also Fig. 12.20) shows that a Hamming window shrinks the passband. This is generally true of tapered windows. To compensate for this shrinkage, we start with a passband somewhat larger (typically 25% larger) than the design passband. In the present case, for instance, selecting $T = 16.67 \mu\text{s}$ would make the passband $\omega_h = \pi/T \approx 2\pi(30,000)$, which is 50% higher than the design passband of 20,000 Hz. ■

⊙ Computer Example C12.10

Using MATLAB, find the frequency response of the 48th-order digital differentiator in Example 12.10. Plot the frequency response for rectangular and Hamming window differentiators.

```

N0=49;
m=(N0-1)/2;
k=0:N0-1;
T=1;
h1=cos((k-m)*pi)/((k-m)*T);h1(25)=0;
num1=h1;
den1=[1, zeros(1,N0-1)];
W=-pi:pi/100:pi;
H1=freqz(num1,den1,W);
mag1=abs(H1);
phase1=180/pi*unwrap(angle(H1));
h2=cos((k-m)*pi)/((k-m)*T).*(0.54+0.46*cos(pi*(k-m)/m));h2(25)=0;

```

```

num2=h2;
den2=den1;
H2=freqz(num2,den2,W);
mag2=abs(H2);
phase2=180/pi*unwrap(angle(H2));
subplot(2,1,1);
plot(W,mag1,W,mag2);
subplot(2,1,2);
plot(W,phase2,W,phase1);

```

△ Exercise E12.7

Design a sixth-order nonrecursive filter to realize $H_a(j\omega) = \Delta(\omega T/\pi)$

Answer:

$$h[k] = \begin{cases} \frac{1}{4} \text{sinc}^2\left(\frac{\pi k}{4}\right) & |k| \leq 3 \\ 0 & |k| > 3 \end{cases} \quad \nabla$$

12.8-2 Nonrecursive Filter Design by the Frequency-Domain Criterion: The Frequency Sampling Method

The frequency-domain criterion is [see Eq. (12.36)]

$$H_a(s) = H[e^{sT}]$$

In this case we shall realize this equality for real frequencies; that is, for $s = j\omega$:

$$H_a(j\omega) = H[e^{j\omega T}] \quad (12.92)$$

For an n th-order filter, there are only $N_0 = n + 1$ elements in $h[k]$, and we can hope to force the two frequency spectra in Eq. (12.92) to be equal only at N_0 points. Because the spectral width is $\frac{2\pi}{T}$, we choose these frequencies $\omega_0 = \frac{2\pi}{N_0 T}$ rad/s apart; that is,

$$\omega_0 = \frac{2\pi}{N_0 T} \quad (12.93)$$

We require that

$$H_a(jr\omega_0) = H[e^{jr\omega_0 T}] \quad r = 0, 1, 2, \dots, N_0 - 1 \quad (12.94)$$

Note that these are the samples of the periodic extension of $H_a(j\omega)$ (or $H[e^{j\omega T}]$). Because we force the frequency response of the filter to be equal to the desired frequency response at N_0 equidistant frequencies in the spectrum, this method is known as the **frequency sampling** or the **spectral sampling** method.

In order to find the filter transfer function, we first determine the filter impulse response $h[k]$. Thus, our problem is to determine the filter impulse response from the knowledge of the N_0 uniform samples of the periodic extension of the filter frequency response $H[e^{j\omega T}]$. But $H[e^{j\omega T}]$ is the DTFT of $h[k]$ [see Eq. (10.92)]. Hence, as shown in Sec. 10.6-2 [Eqs. (10.68) and (10.69)], $h[k]$ and $H[e^{jr\omega_0 T}]$ (the N_0 uniform samples of $H[e^{j\omega T}]$) are the DFT pair with $\Omega_0 = \omega_0 T$. Hence, the desired $h[k]$ is the IDFT of $H[e^{j\omega T}]$, given by

$$\begin{aligned}
 h[k] &= \frac{1}{N_0} \sum_{r=0}^{N_0-1} H[e^{jr\omega_0 T}] e^{jrk\omega_0 T} \\
 &= \frac{1}{N_0} \sum_{r=0}^{N_0-1} H[e^{jr\omega_0 T}] e^{j\frac{2\pi rk}{N_0}} \quad k = 0, 1, 2, \dots, N_0 - 1 \quad (12.95)
 \end{aligned}$$

Note that $H[e^{jr\omega_0 T}] = H_a(jr\omega_0)$ are known [Eq. (12.94)]. We can use IFFT to compute the N_0 values of $h[k]$. From these values of $h[k]$, we can determine the filter transfer function $H[z]$ as

$$H[z] = \sum_{k=0}^{N_0-1} h[k] z^{-k} \quad (12.96)$$

Linear Phase (Constant Delay) Filters

We desire that $H_a(j\omega) = H[e^{j\omega T}]$. The filter featured in Eqs. (12.95) and (12.96) satisfies this condition only at the N_0 values of ω . Between samples, the frequency response, especially the phase response, could deviate considerably. If we want a linear phase characteristic, the procedure is slightly modified. First, we start with a zero phase (or a constant phase $\pm \frac{\pi}{2}$) response. For such a frequency response $h[k]$ is an even (or odd) function of k (see Prob. 4.1-1). In either case, $h[k]$ is centered at $k = 0$. To realize this noncausal filter, we need to delay $h[k]$ by $(N_0 - 1)/2$ units. Such a delay amounts to multiplying $H[e^{j\omega T}]$ with $e^{-j\frac{N_0-1}{2}\omega T}$. Thus, the delay of $h[k]$ does not alter the filter amplitude response, but the phase response changes by $-(N_0 - 1)\omega T/2$, which is a linear function of ω . Hence, we are assured that the filter is realizable (causal) and has a linear phase. Thus, if we wish to realize a frequency response $H[e^{j\omega T}]$, we begin with $H[e^{j\omega T}]e^{-j\frac{N_0-1}{2}\omega T}$ and find the IDFT of its N_0 samples. The resulting IDFT at $k = 0, 1, 2, 3, \dots, N_0 - 1$ is the desired impulse response, which is causal, and the corresponding phase response is linear.

Note that $\omega_0 T = \frac{2\pi}{N_0}$, and the N_0 uniform samples of $H[e^{j\omega T}]e^{-j\frac{N_0-1}{2}\omega T}$ are

$$\begin{aligned}
 H_r &= H[e^{jr\omega_0 T}] e^{-jr\frac{N_0-1}{2}\omega_0 T} \\
 &= H \left[e^{j\frac{2\pi r}{N_0 T}} \right] e^{-jr\pi\frac{N_0-1}{N_0}} \quad (12.97)
 \end{aligned}$$

Recall that the N_0 samples H_r are the uniform samples of the periodic extension of $H[e^{j\omega T}]e^{-j\frac{N_0-1}{2}\omega T}$. Hence, Eq. (12.97) applies to samples of the frequency range from $0 \leq \omega \leq \frac{\pi}{T}$. The remaining samples are obtained by using the conjugate symmetry property $H_r = H_{N_0-r}^*$. The desired $h[k]$ is the IDFT of H_r ; that is,

$$h[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} H_r e^{j\frac{2\pi rk}{N_0}} \quad k = 0, 1, 2, \dots, N_0 - 1 \quad (12.98)$$

and

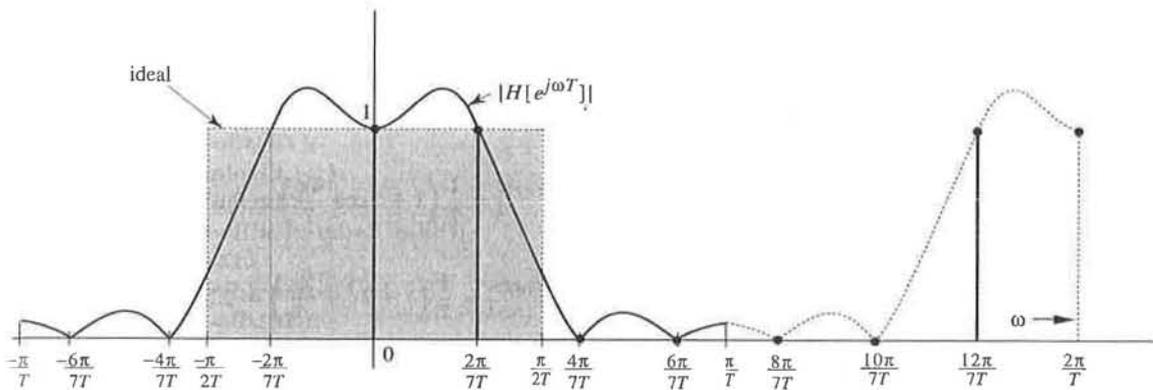


Fig. 12.21 Lowpass filter design using the frequency sampling method.

$$H[z] = \sum_{k=0}^{N_0-1} h[k]z^{-k} \quad (12.99)$$

This procedure will now be explained by an example.

■ **Example 12.12**

Using the frequency sampling method, design a sixth-order nonrecursive ideal lowpass filter of cutoff frequency $\frac{\pi}{2T}$ rad/s.

The frequency response $H[e^{j\omega T}]$ of an ideal lowpass filter is shown (shaded) in Fig. 12.21. In this case

$$N_0 = 7, \quad \frac{N_0 - 1}{N_0} = \frac{6}{7}, \quad \omega_0 = \frac{2\pi}{N_0 T} = \frac{2\pi}{7T}$$

The seven samples H_r in Eq. (12.97) are

$$H_r = H \left[e^{j\frac{2\pi r}{7T}} \right] e^{-j\frac{6\pi r}{7}}$$

Thus

$$H_0 = 1, \quad H_1 = e^{-j\frac{6\pi}{7}}, \quad H_2 = H_3 = 0$$

The remaining three samples should be determined using the conjugate property of DFT, $H_r = H_{N_0-r}^*$, that is $H_r = H_{7-r}^*$. Thus

$$H_4 = H_3^* = 0, \quad H_5 = H_2^* = 0, \quad H_6 = H_1^* = e^{j\frac{6\pi}{7}}$$

The desired $h[k]$ is the IDFT of H_r given by [Eq. (12.98)]

$$h[k] = \sum_{r=0}^6 H_r e^{j\frac{2\pi r k}{7}} \quad k = 0, 1, 2, 3, \dots, 6$$

We may compute this IDFT by using the IFFT algorithm or by straightforward substitution of values of H_r in the above equation as

$$h[0] = \frac{1}{7} \sum_{r=0}^6 H_r = \frac{1}{7} [1 + e^{-j6\pi/7} + e^{j6\pi/7}] = \frac{1}{7} \left(1 + 2 \cos \frac{6\pi}{7} \right) = -0.1146$$

$$h[1] = \frac{1}{7} \sum_{r=0}^6 H_r e^{j\frac{2\pi r}{7}} = \frac{1}{7} [1 + e^{-j4\pi/7} + e^{j4\pi/7}] = \frac{1}{7} \left(1 + 2 \cos \frac{4\pi}{7} \right) = 0.0792$$

$$h[2] = \frac{1}{7} \sum_{r=0}^6 H_r e^{j\frac{4\pi r}{7}} = \frac{1}{7} [1 + e^{-j2\pi/7} + e^{j2\pi/7}] = \frac{1}{7} \left(1 + 2 \cos \frac{2\pi}{7} \right) = 0.3209$$

$$h[3] = \frac{1}{7} \sum_{r=0}^6 H_r e^{j\frac{6\pi r}{7}} = \frac{1}{7} [1 + 1 + 1] = 0.4285$$

Similarly, we can show that

$$h[4] = 0.3209, \quad h[5] = 0.0792, \quad \text{and} \quad h[6] = -0.1146$$

Observe that $h[k]$ is symmetrical about its center point $k = 3$ as expected. Compare these values with those found by the impulse invariance method in Table 12.3 for a rectangular window. Although the two sets of values are different, they are comparable. What is the difference in the two filters? The impulse invariance filter optimizes the design with respect to all frequencies. It minimizes the mean squared value of the difference between the desired and the realized frequency response. The frequency sampling method, in contrast, realizes a filter whose frequency response matches exactly to the desired frequency response at N_0 uniformly spaced frequencies. The mean squared error in this design will generally be higher than that in the impulse invariance method.

The filter transfer function is

$$\begin{aligned} H[z] &= -0.1146 + \frac{0.0792}{z} + \frac{0.3209}{z^2} + \frac{0.4285}{z^3} + \frac{0.3209}{z^4} + \frac{0.0792}{z^5} - \frac{0.1146}{z^6} \\ &= \frac{-0.1146z^6 + 0.0792z^5 + 0.3209z^4 + 0.4285z^3 + 0.3209z^2 + 0.0792z - 0.1146}{z^6} \end{aligned}$$

and

$$\begin{aligned} H[e^{j\omega T}] &= -0.1146 + 0.0792e^{-j\omega T} + 0.3209e^{-j2\omega T} + 0.4285e^{-j3\omega T} \\ &\quad + 0.3209e^{-j4\omega T} + 0.0792e^{-j5\omega T} - 0.1146e^{-j6\omega T} \\ &= e^{-j3\omega T} [0.4285 + 0.6418 \cos \omega T + 0.1584 \cos 2\omega T - 0.2292 \cos 3\omega T] \end{aligned}$$

The magnitude of this response (Fig. 12.21) shows that the realized filter values match exactly the desired response at the N_0 sample points. The time delay adds a linear phase $-3T\omega$ to the filter characteristic. ■

⊙ Computer Example C12.11

Using MATLAB, find $h[k]$ and the corresponding $H[e^{j\omega T}]$ for the frequency sampling filter in Example 12.12.

```
N0=7;
H=[1 1 0 0 0 0 1];
for i=1:N0
```

```

r=i-1;
Hr(i)=H(i)*exp(-j*r*pi*(N0-1)/N0);
end
k=0:6
hk=ifft(Hr);
subplot(2,1,1);
stem(k,hk);
xlabel('k');ylabel('h[k]');
M=512
hE=[hk zeros(1,M-7)]
HE=fft(hE);
subplot(2,1,2);
r=0:M-1;
W=r.*2*pi/512
plot(W,abs(HE));
xlabel('W');ylabel('F(W)');grid;

```

An Alternate Method Using Frequency Sampling Filters

We now show an alternative approach to the frequency sampling method, which uses an N_0 -order comb filter in cascade with a parallel bank of $N_0 - 1$ first-order filters. This structure forms a frequency sampling filter. We start with Eqs. (6.55). The transfer function $H[z]$ of the filter is first obtained by taking the z -transform of $h[k]$ in Eq. (12.98):

$$\begin{aligned}
 H[z] &= \sum_{k=0}^{N_0-1} h[k]z^{-k} \\
 &= \frac{1}{N_0} \sum_{k=0}^{N_0-1} \left[\sum_{r=0}^{N_0-1} H_r e^{j\frac{2\pi kr}{N_0}} \right] z^{-k} \\
 &= \frac{1}{N_0} \sum_{r=0}^{N_0-1} H_r \sum_{k=0}^{N_0-1} (e^{j\frac{2\pi r}{N_0}} z^{-1})^k
 \end{aligned}$$

The second sum on the right-hand side is a geometric series, and using the result in B.7-4, we have

$$\begin{aligned}
 H[z] &= \frac{1}{N_0} \sum_{r=0}^{N_0-1} H_r \left[\frac{1 - z^{-N_0} e^{j2\pi r}}{1 - e^{j\frac{2\pi r}{N_0}} z^{-1}} \right] \\
 &= \frac{1}{N_0} \sum_{r=0}^{N_0-1} H_r \left[\frac{z^{-N_0} (z^{N_0} - 1)}{z^{-1} (z - e^{j\frac{2\pi r}{N_0}})} \right]
 \end{aligned}$$

Hence,

$$H[z] = \underbrace{\frac{z^{N_0} - 1}{N_0 z^{N_0}}}_{H_1[z]} \underbrace{\sum_{r=0}^{N_0-1} \frac{z H_r}{z - e^{j\frac{2\pi r}{N_0}}}}_{H_2[z]} \quad (12.100)$$

Observe that we do not need to perform IDFT (or IFFT) computations to obtain the desired filter transfer function. All we need is the values of the frequency samples H_r , which are given. Equation (12.100) shows that the desired filter is realized as a cascade of two filters with transfer functions $H_1[z]$ and $H_2[z]$. Also, $H_1[z]$ is the transfer function of an N_0 -order comb filter (see Example 12.9). The second filter with transfer function $H_2[z]$ is a parallel combination of $n + 1$ first-order filters, whose poles lie on the unit circle at $e^{j\frac{2\pi r}{N_0}}$ ($r = 0, 1, 2, \dots, N_0 - 1$). For the lowpass or bandpass filters many coefficients H_r appearing in $H_2[z]$ are zero. Recall that in Example 12.12 (lowpass filter) four out of seven coefficients are zero. Thus, in practice the final filter is usually much simpler than it appears in Eq. (12.100). As a result, this method may require a fewer number of computations (multiplications and additions) compared to those in the filter obtained by the direct method (using IDFT).

The poles of the frequency sampling filter are complex in general because they lie on the unit circle. Therefore, we must combine the conjugate poles to form quadratic transfer functions to realize them. All these points will be clarified by designing the filter in Example 12.12 by this method.

■ Example 12.13

Redo Example 12.12 using the method of frequency sampling filter.

In this case $n = 6$, $N_0 = n + 1 = 7$. Also

$$H_2 = H_3 = H_4 = H_5 = 0$$

and

$$H_0 = 1, H_1 = e^{-j6\pi/7}, H_6 = e^{j6\pi/7}$$

Substituting these values in the transfer function $H[z]$ of the desired filter in Eq. (12.100), we obtain

$$H[z] = \underbrace{\frac{z^7 - 1}{7z^7}}_{H_1[z]} \underbrace{\left[\frac{z}{z-1} + \frac{ze^{-j6\pi/7}}{z - e^{j2\pi/7}} + \frac{ze^{j6\pi/7}}{z - e^{-j2\pi/7}} \right]}_{H_2[z]}$$

We combine the last two terms on the right-hand side corresponding to complex conjugate poles to obtain

$$\begin{aligned} H[z] &= \frac{z^7 - 1}{7z^7} \left[\frac{z}{z-1} + \frac{z(2z \cos \frac{6\pi}{7} - 2 \cos \frac{8\pi}{7})}{z^2 - (2 \cos \frac{2\pi}{7} z + 1)} \right] \\ &= \underbrace{\frac{z^7 - 1}{7z^7}}_{H_1[z]} \underbrace{\left[\frac{z}{z-1} + \frac{1.802z(z-1)}{z^2 - 1.247z + 1} \right]}_{H_2[z]} \end{aligned}$$

We can realize this filter by placing the comb filter $H_1[z]$ in cascade with $H_2[z]$, which consists of a first-order and a second-order filter in parallel. ■

Pole-Zero Cancellation in Frequency Sampling Filters

In the method of frequency sampling filters, we make an intriguing observation that the required nonrecursive (FIR) filter is realized by a cascade of $H_1[z]$ and

$H_2[z]$. However, $H_2[z]$ is recursive (IIR). This strange fact should alert us to the possibility of something interesting going on in this filter. For a nonrecursive filter there can be no poles other than those at the origin [see Eq. (12.69a)]. In the frequency sampling filter [Eq. (12.100)], in contrast, $H_2[z]$ has N_0 poles at $e^{jr\Omega_0}$ ($r = 0, 1, 2, \dots, n$). All these poles lie on the unit circle at equally spaced points. These poles simply *cannot be* in a nonrecursive filter. They must somehow get canceled along the way somewhere. This is precisely what happens. The zeros of $H_1[z]$ are exactly where the poles of $H_2[z]$ are because

$$z^{N_0} - 1 = (z - e^{j0\frac{2\pi}{N_0}})(z - e^{j\frac{2\pi}{N_0}})(z - e^{j2\frac{2\pi}{N_0}})\dots(z - e^{jn\frac{2\pi}{N_0}})$$

Thus, the poles of $H_2[z]$ are canceled by the zeros of $H_1[z]$, rendering the final filter nonrecursive.

Pole-zero cancellation in this filter is a potential cause for mischief because such a perfect cancellation assumes exact realization of both $H_1[z]$ and $H_2[z]$. Such a realization requires the use of infinite precision arithmetic, which is a practical impossibility because of quantization effects. Imperfect cancellation of poles and zeros means there will still be poles on the unit circle, and the filter will not have a finite impulse response. More serious, however, is the fact that the resulting system will be marginally stable. Such a system provides no damping of the round-off noise that is introduced in the computations. In fact, such noise tends to increase with time, and may render the filter useless.

We can partially mitigate this problem by moving both the poles (of $H_2[z]$) and zeros (of $H_1[z]$) to a circle of radius $r = 1 - \epsilon$, where ϵ is a small positive number $\rightarrow 0$. This artifice will make the overall system asymptotically stable.

Spectral Sampling with Windowing

The frequency sampling method can be modified to take advantage of windowing. We first design a frequency sampling filter using a value N_0' that is much higher than the design value N_0 . The result is a filter that matches with the desired frequency response at a very large number (N_0') of points. Then we use a suitable N_0 -point window to truncate the N_0' -point impulse response. This procedure yields the final design of a desired order.

12.9 Summary

The response of an LTID system with transfer function $H[z]$ to an everlasting sinusoid of frequency Ω is also an everlasting sinusoid of the same frequency. The output amplitude is $|H[e^{j\Omega}]|$ times the input amplitude, and the output sinusoid is shifted in phase with respect to the input sinusoid by $\angle H[e^{j\Omega}]$ radians. The plot of $|H[e^{j\Omega}]|$ vs Ω indicates the amplitude gain of sinusoids of various frequencies and is called the *amplitude response* of the system. The plot of $\angle H[e^{j\Omega}]$ vs Ω indicates the phase shift of sinusoids of various frequencies and is called the *phase response*. The frequency response of an LTID system is a periodic function of Ω with period 2π . This periodicity is the result of the fact that discrete-time sinusoids with frequencies differing by an integral multiple of 2π are identical.

Frequency response of a system is determined by locations in the complex plane of poles and zeros of its transfer function. We can design frequency selective filters by

proper placement of its transfer function poles and zeros. Placing a pole (or a zero) near the point $e^{j\Omega_0}$ in the complex plane enhances (or suppresses) the frequency response at the frequency $\Omega = \Omega_0$. Using this concept, a proper combination of poles and zeros at suitable locations can yield desired filter characteristics.

Digital filters are classified into recursive and nonrecursive filters. The duration of the impulse response of a recursive filter is infinite; that of the nonrecursive filter is finite. For this reason, recursive filters are also called infinite impulse response (IIR) filters, and nonrecursive filters are called finite impulse response (FIR) filters.

Digital filters can process analog signals using A/D and D/A converters. Procedures for designing a digital filter that behaves like a given analog filter are discussed. A digital filter can simulate the behavior of a given analog filter either in time-domain or in frequency-domain. This situation leads to two different design procedures, one using a time-domain equivalence criterion and the other a frequency-domain equivalence criterion.

For recursive or IIR filters, the time-domain equivalence criterion yields the impulse invariance method, and the frequency-domain equivalence criterion yields the bilinear transformation method. The impulse invariance method is handicapped by the aliasing problem, and cannot be used for highpass and bandstop filters. The bilinear transformation method, which is generally superior to the impulse invariance method, suffers from the frequency scale warping effect. However, this effect can be neutralized by prewarping.

For nonrecursive or FIR filters, the time-domain equivalence criterion leads to the method of windowing (Fourier series method), and the frequency-domain equivalence criterion leads to the method of frequency sampling. Because nonrecursive filters are a special case of recursive filters, we expect the performance of recursive filters to be superior. This statement is true in the sense that a given amplitude response can be achieved by a recursive filter of an order smaller than that required for the corresponding nonrecursive filter. However, nonrecursive filters have the advantage that it can realize an arbitrarily shaped amplitude response and a linear phase characteristic. Recursive filters are good for a piecewise constant amplitude response and they can realize linear phase only approximately. To realize a linear phase characteristic in nonrecursive filters, the impulse response $h[k]$ must be either symmetric or antisymmetric about its center point.

References

1. Mitra, S.K., *Digital Signal processing: A Computer-Based Approach*, McGraw-Hill, New York, 1998.

Problems

- 12.1-1 Find the amplitude and phase response of the digital filters depicted in Fig. P12.1-1.
- 12.1-2 Find the amplitude and the phase response of the filters shown in Fig. P12.1-2.
Hint: Express $H[e^{j\Omega}]$ as $e^{-j2.5\Omega} H_a[e^{j\Omega}]$.
- 12.1-3 For an LTID system specified by the equation

$$y[k+1] - 0.5y[k] = f[k+1] + 0.8f[k]$$

- (a) Find the amplitude and the phase response.
- (b) Find the system response $y[k]$ for the input $f[k] = \cos(0.5k - \frac{\pi}{3})$.

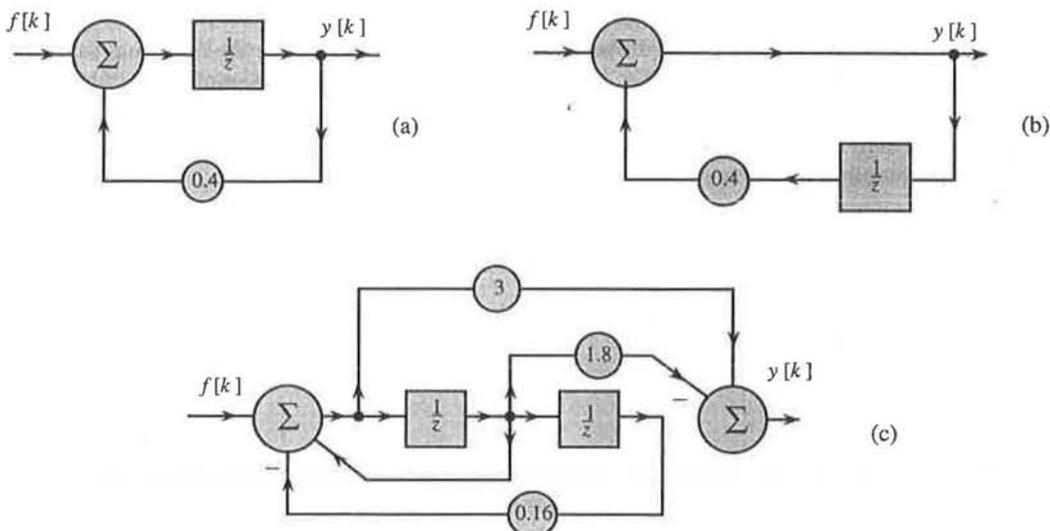


Fig. P12.1-1.

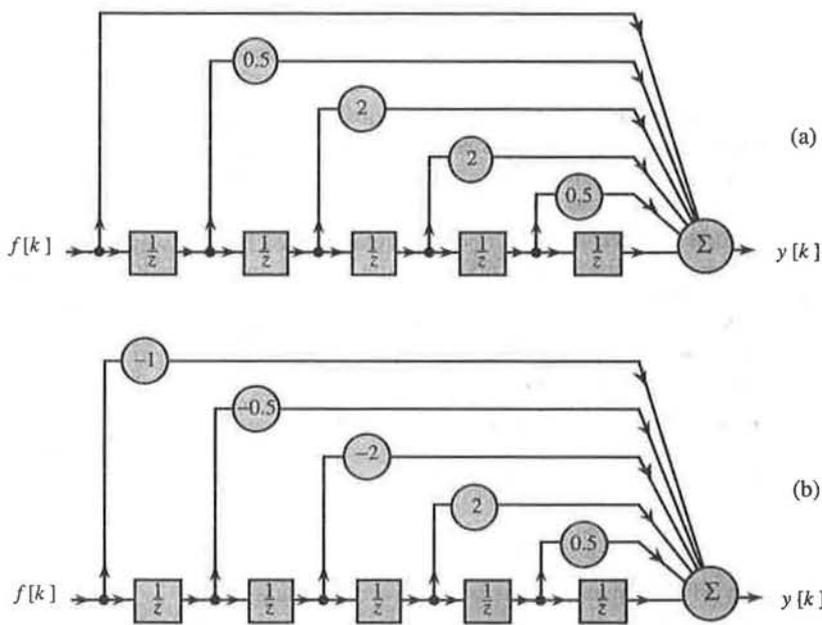


Fig. P12.1-2.

- 12.1-4 For an asymptotically stable LTID system, show that the steady-state response to input $e^{j\Omega k} u[k]$ is $H[e^{j\Omega}] e^{j\Omega k} u[k]$. The steady-state response is that part of the response which does not decay with time and persists forever.
 Hint: Follow the procedure parallel to that used for continuous-time systems in Sec. 7.1-1.
- 12.1-5 (a) A digital filter has the sampling interval $T = 50 \mu\text{s}$. Determine the highest