

The counterpart of the Laplace transform for discrete-time systems is the z-transform. The Laplace transform converts integro-differential equations into algebraic equations. In the same way, the z-transforms changes difference equations into algebraic equations, thereby simplifying the analysis of discrete-time systems. The z-transform method of analysis of discrete-time systems parallels the Laplace transform method of analysis of continuous-time systems, with some minor differences. In fact, we shall see that the z-transform is the Laplace transform in disguise.

The behavior of discrete-time systems (with some differences) is similar to that of continuous-time systems. The frequency-domain analysis of discrete-time systems is based on the fact (proved in Sec. 9.4-2) that the response of a linear time-invariant discrete-time (LTID) system to an everlasting exponential z^k is also the same exponential (within a multiplicative constant), given by $H[z]z^k$. We then express an input f[k] as a sum of (everlasting) exponentials of the form z^k . The system response to f[k] is then found as a sum of the system's responses to all these exponential components. The tool which allows us to represent an arbitrary input f[k] as a sum of (everlasting) exponentials of the z-transform.

11.1 The Z-Transform

In the last Chapter, we extended the discrete-time Fourier transform to derive the pair of equations defining the z-transform as

$$F[z] \equiv \sum_{k=-\infty}^{\infty} f[k] z^{-k}$$
(11.1)

$$f[k] = \frac{1}{2\pi j} \oint F[z] z^{k-1} dz$$
 (11.2)

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where the symbol \oint indicates an integration in counterclockwise direction around a closed path in the complex plane (see Fig. 11.1). As in the case of the Laplace transform, we need not worry about this integral at this point because inverse ztransforms of many signals of engineering interest can be found in a z-transform Table. The direct and inverse z-transforms can be expressed symbolically as

$$F[z] = \mathcal{Z} \{ f[k] \} \quad \text{and} \quad f[k] = \mathcal{Z}^{-1} \{ F[z] \}$$

or simply as

$$f[k] \Longleftrightarrow F[z]$$

Note that

$$\mathcal{Z}^{-1}\left[\mathcal{Z}\left\{f[k]\right\}\right] = f[k] \quad \text{and} \quad \mathcal{Z}\left[\mathcal{Z}^{-1}\left\{F[z]\right\}\right] = F[z]$$

Following the earlier argument, we can find an LTID system response to an input f[k] using the steps as follows:

$$z^k \implies H[z]z^k$$
 the system response to z^k is $H[z]z^k$

$$f[k] = \frac{1}{2\pi j} \oint F[z] z^{k-1} dz \quad \text{shows } f[k] \text{ as a sum of everlasting exponential components}$$

and

$$y[k] = \frac{1}{2\pi j} \oint F[z]H[z]z^{k-1} dz \text{ shows } y[k] \text{ as a sum of responses to exponential components}$$
$$= \frac{1}{2\pi j} \oint Y[z]z^{k-1} dz$$

where

$$Y[z] = F[z]H[z]$$

In conclusion, we have shown that for an LTID system with transfer function H[z], if the input and the output are f[k] and y[k], respectively, and if

$$f[k] \iff F[z] \qquad y[k] \iff Y[z]$$

then

$$Y[z] = F[z]H[z]$$

We shall derive this result more formally later.

Linearity of the Z-Transform

Like the Laplace transform, the z-transform is a linear operator. If

$$f_1[k] \iff F_1[z]$$
 and $f_2[k] \iff F_2[z]$

then

$$a_1f_1[k] + a_2f_2[k] \iff a_1F_1[z] + a_2F_2[z]$$
 (11.3)

The proof is trivial and follows from the definition of the z-transform. This result can be extended to finite sums.

The Unilateral Z-Transform

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For the same reasons discussed in Chapter 6, we first start with a simpler version of the z-transform, the **unilateral** z-transform, that is restricted only to the analysis of causal systems with causal inputs (signals starting at k = 0). The more general bilateral z-transform is discussed later in Sec. 11.7. In the unilateral case, the signals are restricted to be causal; that is, they start at k = 0. The definition of the unilateral transform is the same as that of the bilateral [Eq. (11.1)] except that the limits of the sum are from 0 to ∞

$$F[z] \equiv \sum_{k=0}^{\infty} f[k] z^{-k} \tag{11.4}$$

where z is complex in general. The expression for the inverse z-transform in Eq. (11.2) remains valid for the unilateral case also.

The Region of Convergence of $\mathbf{F}[\mathbf{z}]$

The sum in Eq. (11.1) [or (11.4)] defining the direct z-transform F[z] may not converge (exist) for all values of z. The values of z (the region in the complex plane) for which the sum in Eq. (11.1) converges (or exists) is called the **region of convergence** (or **region of existence**) of F[z]. This concept will become clear in the following example.

Example 11.1

Find the z-transform and the corresponding region of convergence for the signal $\gamma^k u[k].$

By definition

$$F[z] = \sum_{k=0}^{\infty} \gamma^k u[k] z^{-k}$$

Since u[k] = 1 for all $k \ge 0$,

$$F[z] = \sum_{k=0}^{\infty} \left(\frac{\gamma}{z}\right)^{k}$$
$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^{2} + \left(\frac{\gamma}{z}\right)^{3} + \dots + \dots$$
(11.5)

It is helpful to remember the following well-known geometric progression and its sum:

$$1 + x + x^{2} + x^{3} + \dots = \frac{1}{1 - x}$$
 if $|x| < 1$ (11.6)

Use of Eq. (11.6) in Eq. (11.5) yields

$$F[z] = \frac{1}{1 - \frac{\gamma}{z}} \qquad \left| \frac{\gamma}{z} \right| < 1$$
$$= \frac{z}{z - \gamma} \qquad |z| > |\gamma| \qquad (11.7)$$



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Fig. 11.1 $\gamma^k u[k]$ and the region of convergence of its z-transform.

Observe that F[z] exists only for $|z| > |\gamma|$. For $|z| < |\gamma|$, the sum in Eq. (11.5) does not converge; it goes to infinity. Therefore, the region of convergence (or existence) of F[z] is the shaded region outside the circle of radius $|\gamma|$, centered at the origin, in the z-plane, as depicted in Fig. 11.1b.

The region of convergence is required for evaluating f[k] from F[z], according to Eq. (11.2). The integral in Eq. (11.2) is a contour integral implying integration in a counterclockwise direction along a closed path centered at the origin and satisfying the condition $|z| > |\gamma|$. Thus, any circular path centered at the origin and with a radius greater than $|\gamma|$ (Fig. 11.1b) will suffice. We can show that the integral in Eq. (11.2) along any such path (with a radius greater than $|\gamma|$) yields the same result, namely f[k]. Such integration in the complex plane requires a background in the theory of functions of complex variables. We can avoid this integration by compiling a table of z-transforms (Table 11.1), where z-transform pairs are tabulated for a variety of signals. To find the inverse z-transform of say, $z/(z - \gamma)$, instead of using the complex integration in (11.2), we consult the table and find the inverse z-transform of $z/(z - \gamma)$ as $\gamma^k u[k]$. Although the table given here is rather short, it comprises the functions of most practical integrat.

The bilateral z-transform is defined by Eq. (11.1) with the limits of the righthand sum from $-\infty$ to ∞ instead of from 0 to ∞ . The situation of the z-transform regarding the uniqueness of the inverse transform is parallel to that of the Laplace transform. For the bilateral case, the inverse z-transform is not unique unless the region of convergence is specified. For the unilateral case, the inverse transform is unique; the region of convergence need not be specified to determine the inverse ztransform. For this reason, we shall ignore the region of convergence in the unilateral z-transform Table 11.1.

Existence of the Z-Transform

By definition

$$F[z] = \sum_{k=0}^{\infty} f[k] z^{-k} = \sum_{k=0}^{\infty} \frac{f[k]}{z^k}$$



The existence of the z-transform is guaranteed if

$$|F[z]| \leq \sum_{k=0}^{\infty} \frac{|f[k]|}{|z|^k} < \infty$$

for some |z|. Any signal f[k] that grows no faster than an exponential signal r_0^k , for some r_0 , satisfies this condition. Thus, if

 $|f[k]| \le {r_0}^k \qquad \text{for some } r_0 \tag{11.8}$

then

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$$|F[z]| \leq \sum_{k=0}^{\infty} \left(\frac{r_0}{|z|}\right)^k = \frac{1}{1 - \frac{r_0}{|z|}} \qquad |z| > r_0$$

Therefore, F[z] exists for $|z| > r_0$. All practical signals satisfy (11.8) and are therefore z-transformable. Some signal models (e.g. γ^{k^2}) which grow faster than the exponential signal r_0^k (for any r_0) do not satisfy (11.8) and therefore are not z-transformable. Fortunately, such signals are of little practical or theoretical interest.

Example 11.2

Find the z-transforms of (a) $\delta[k]$ (b) u[k] (c) $\cos \beta k u[k]$ (d) signal shown in Fig. 11.2.

Recall that by definition

$$F[z] = \sum_{k=0}^{\infty} f[k] z^{-k}$$

= $f[0] + \frac{f[1]}{z} + \frac{f[2]}{z^2} + \frac{f[3]}{z^3} + \cdots$ (11.9)

(a) For $f[k] = \delta[k]$, f[0] = 1 and $f[2] = f[3] = f[4] = \cdots = 0$. Therefore

$$S[k] \iff 1$$
 for all z (11.10)

(b) For f[k] = u[k], $f[0] = f[1] = f[3] = \cdots = 1$. Therefore

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$$F[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

From Eq. (11.6) it follows that

$$F[z] = \frac{1}{1 - \frac{1}{z}} \qquad \left| \frac{1}{z} \right| < 1$$
$$= \frac{z}{z - 1} \qquad |z| > 1$$

Therefore

$$[k] \Longleftrightarrow \frac{z}{z-1} \qquad |z| > 1 \tag{11.11}$$

(c) Recall that $\cos \beta k = \left(e^{j\beta k} + e^{-j\beta k}\right)/2$. Moreover, according to Eq. (11.7),

$$e^{\pm j\beta k}u[k] \iff rac{z}{z-e^{\pm j\beta}} \qquad |z|>|e^{\pm j\beta}|=1$$

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Therefore

$$F[z] = \frac{1}{2} \left[\frac{z}{z - e^{j\beta}} + \frac{z}{z - e^{-j\beta}} \right]$$
$$= \frac{z(z - \cos\beta)}{z^2 - 2z\cos\beta + 1} \qquad |z| > 1$$

(d) Here f[0] = f[1] = f[2] = f[3] = f[4] = 1 and $f[5] = f[6] = \cdots = 0$. Therefore, according to Eq. (11.9)

$$F[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$$
$$= \frac{z^4 + z^3 + z^2 + z + 1}{z^4}$$

We can also express this result in a closed form by summing the geometric progression on the right-hand side of the above equation, using the formula in Sec. B.7-4. Here the common ratio $r = \frac{1}{z}$, M = 0, and N = 4, so that



 \triangle Exercise E11.1

(a) Find the z-transform of a signal shown in Fig. 11.3. (b) Using Pair 12a (Table 11.1) find the z-transform of $f[k] = 20.65(\sqrt{2})^k \cos\left(\frac{\pi}{4}k - 1.415\right) u[k]$.

Answers: (a)
$$F[z] = \frac{z^5 + z^4 + +z^3 + z^2 + z + 1}{z^9}$$
 or $\frac{z}{z-1}(z^{-4} - z^{-10})$
(b) $\frac{z(3.2z + 17.2)}{z^2 - 2z + 2}$ ∇

11.1-1 Finding the Inverse Transform

As in the Laplace transform, we shall avoid the integration in the complex plane required to find the inverse z-transform [Eq. (11.2)] by using the (unilateral) transform Table. Many of the transforms F[z] of practical interest are rational functions (ratio of polynomials in z). Such functions can be expressed as a sum of simpler functions using partial fraction expansion. This method works because for every transformable f[k] defined for $k \ge 0$, there is a corresponding unique F[z]defined for $|z| > r_0$ (where r_0 is some constant), and vice versa.



f[/	٤]	F[z]
1	$\delta[k-j]$	z^{-j}
2	u[k]	$\frac{z}{z-1}$
3	ku[k]	$rac{z}{(z-1)^2}$
4	$k^2 u[k]$	$\frac{z(z+1)}{(z-1)^3}$
5	$k^3u[k]$	$\frac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^{k-1}u[k-1]$	$rac{1}{z-\gamma}$
7	$\gamma^k u[k]$	$rac{z}{z-\gamma}$
8	$k\gamma^k u[k]$	$rac{\gamma z}{(z-\gamma)^2}$
9	$k^2 \gamma^k u[k]$	$rac{\gamma z(z+\gamma)}{(z-\gamma)^3}$
10	$\frac{k(k-1)(k-2)\cdots(k-m+1)}{\gamma^m m!}\gamma^k u[$	$k] \qquad \frac{z}{(z-\gamma)^{m+1}}$
11a	$ \gamma ^k \cos \beta k u[k]$	$\frac{z(z- \gamma \cos\beta)}{z^2-(2 \gamma \cos\beta)z+ \gamma ^2}$
11b	$ \gamma ^k \sin \beta k u[k]$	$\frac{z \gamma \sin\beta}{z^2-(2 \gamma \cos\beta)z+ \gamma ^2}$
12a	$r \gamma ^k\cos{(\beta k+ heta)u[k]}$	$\frac{rz[z\cos\theta- \gamma \cos\left(\beta-\theta\right)]}{z^2-(2 \gamma \cos\beta)z+ \gamma ^2}$
12b	$r \gamma ^k\cos{(eta k+ heta)u[k]} \qquad \gamma= \gamma e^{jeta}$	$\frac{(0.5re^{j\theta})z}{z-\gamma}+\frac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r \gamma ^k\cos{(\beta k+ heta)u[k]}$	$\frac{z(Az+B)}{z^2+2az+ \gamma ^2}$
	$r = \sqrt{\frac{A^2 \gamma ^2 + B^2 - 2AaB}{ \gamma ^2 - a^2}}$	
	$\beta = \cos^{-1} \frac{-a}{ \gamma }, \ \theta = \tan^{-1} \frac{Aa-B}{A\sqrt{ \gamma ^2 - a^2}}$	

Table 11.1: (Unilateral) z-Transform Pairs

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Example 11.3

Find the inverse z-transform of

(a)
$$\frac{8z-19}{(z-2)(z-3)}$$
 (b) $\frac{z(2z^2-.11z+12)}{(z-1)(z-2)^3}$ (c) $\frac{2z(3z+17)}{(z-1)(z^2-6z+25)}$

(a) Expanding F[z] into partial fractions yields

$$F[z] = \frac{8z - 19}{(z - 2)(z - 3)} = \frac{3}{z - 2} + \frac{5}{z - 3}$$

From Table 11.1, Pair 6, we obtain

$$f[k] = \left[3(2)^{k-1} + 5(3)^{k-1}\right] u[k-1]$$
(11.12a)

If we expand rational F[z] into partial fractions directly, we shall always obtain an answer that is multiplied by u[k-1] because of the nature of Pair 6 in Table 11.1. This form is rather awkward as well as inconvenient. We prefer the form that is multiplied by u[k]rather than u[k-1]. A glance at Table 11.1 shows that the z-transform of every signal that is multiplied by u[k] has a factor z in the numerator. This observation suggests that we expand F[z] into modified partial fractions, where each term has a factor z in the numerator. This goal can be accomplished by expanding F[z]/z into partial fractions and then multiplying both sides by z. We shall demonstrate this procedure by reworking part (a) in Example 11.3. For this case

$$\frac{F[z]}{z} = \frac{8z - 19}{z(z - 2)(z - 3)}$$
$$= \frac{(-19/6)}{z} + \frac{(3/2)}{z - 2} + \frac{(5/3)}{z - 3}$$

Multiplying both sides by z yields

$$F[z] = -\frac{19}{6} + \frac{3}{2} \left(\frac{z}{z-2}\right) + \frac{5}{3} \left(\frac{z}{z-3}\right)$$

From Pairs 1 and 7 in Table 11.1, it follows that

$$f[k] = -\frac{19}{6}\delta[k] + \left[\frac{3}{2}(2)^k + \frac{5}{3}(3)^k\right]u[k]$$
(11.12b)

The reader can verify that this answer is equivalent to that in Eq. (11.12a) by computing f[k] in both cases for $k = 0, 1, 2, 3, \cdots$, and then comparing the results. The form in Eq. (11.12b) is more convenient than that in Eq. (11.12a). For this reason, we shall always expand F[z]/z rather than F[z] into partial fractions and then multiply both sides by z to obtain modified partial fractions of F[z], which have a factor z in the numerator.

$$F[z] = \frac{z(2z^2 - 11z + 12)}{(z - 1)(z - 2)^3}$$

and

$$\begin{aligned} \frac{F[z]}{z} &= \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} \\ &= \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)} \end{aligned}$$



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where

$$k = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} \bigg|_{z=1} = -3$$
$$a_0 = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} \bigg|_{z=2} = -2$$

Therefore

$$\frac{F[z]}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$
(11.13)

We can determine a_1 and a_2 by clearing fractions or by using the short cuts discussed in Sec. B.5-3. For example, to determine a_2 , we multiply both sides of Eq. (11.13) by z and let $z \to \infty$. This yields

$$0 = -3 - 0 + 0 + a_2 \Longrightarrow a_2 = 3$$

This result leaves only one unknown, a_1 , which is readily determined by letting z take any convenient value, say z = 0, on both sides of Eq. (11.13). This step yields

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2}$$

Multiplying both sides by 8 yields

$$12 = 24 + 2 + 2a_1 - 12 \Longrightarrow a_1 = -1$$

Therefore

$$\frac{F[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$$

and

$$F[z] = -3\frac{z}{z-1} - 2\frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3\frac{z}{z-2}$$

Now the use of Table 11.1, Pairs 7 and 10, yields

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$$f[k] = \left[-3 - 2\frac{k(k-1)}{8}(2)^k - \frac{k}{2}(2)^k + 3(2)^k\right]u[k]$$
$$= -[3 + \frac{1}{4}(k^2 + k - 12)2^k]u[k]$$

(c) Complex Poles

$$F[z] = \frac{2z(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

Poles of F[z] are 1, 3 + j4, and 3 - j4. Whenever there are complex conjugate poles, the problem can be worked out in two ways. In the first method we expand F[z] into (modified) first-order partial fractions. In the second method, rather than obtaining one factor corresponding to each complex conjugate pole, we obtain quadratic factors corresponding to each pair of complex conjugate poles. This procedure is explained below.

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Method of First-Order Factors

$$\frac{F[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

We find the partial fraction of F[z]/z using the Heaviside "cover-up" method:

$$\frac{F[z]}{z} = \frac{2}{z-1} + \frac{1.6e^{-j2.246}}{z-3-j4} + \frac{1.6e^{j2.246}}{z-3+j4}$$

and

$$F[z] = 2\frac{z}{z-1} + (1.6e^{-j2.246})\frac{z}{z-3-j4} + (1.6e^{j2.246})\frac{z}{z-3+j4}$$

The inverse transform of the first term on the right-hand side is 2u[k]. The inverse transform of the remaining two terms (complex conjugate poles) can be obtained from Pair 12b (Table 11.1) by identifying $\frac{r}{2} = 1.6$, $\theta = -2.246$ rad., $\gamma = 3 + j4 = 5e^{j0.927}$, so that $|\gamma| = 5$, $\beta = 0.927$. Therefore

$$f[k] = \left[2 + 3.2(5)^k \cos\left(0.927k - 2.246\right)\right] u[k]$$

Method of Quadratic Factors

$$\frac{F[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

Multiplying both sides by z and letting $z \to \infty$, we find

$$0 = 2 + A \Longrightarrow A = -2$$

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

To find B, we let z take any convenient value, say z = 0. This step yields

$$\frac{-34}{25} = -2 + \frac{B}{25}$$

Multiplying both sides by 25 yields

$$-34 = -50 + B \Longrightarrow B = 16$$

Therefore

$$\frac{F[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2 - 6z + 25}$$

$$F[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2 - 6z + 25}$$

We now use Pair 12c where we identify A = -2, B = 16, $|\gamma| = 5$, a = -3. Therefore

$$r = \sqrt{\frac{100+256-192}{25-9}} = 3.2, \quad \beta = \cos^{-1}(\frac{3}{5}) = 0.927 \text{ rad., and}$$
$$\theta = \tan^{-1}(\frac{-10}{-8}) = -2.246 \text{ rad., so that}$$
$$f[k] = \left[2 + 3.2(5)^k \cos\left(0.927k - 2.246\right)\right] u[k] \quad \blacksquare$$

and

The procedure for finding partial fractions using MATLAB was demonstrated in chapter 6. The same program can be used in this case, except that we have to find the modified partial fractions here. This goal is readily accomplished by dividing F[z] by z and then taking the partial fractions. We shall demonstrate this procedure with an example.

 \odot Computer Example C11.1 Solve Example 11.3a using MATLAB.

> num=[8 -19]; den=[conv([1 -2],[1 -3]) 0]; [r, p, k]= residue(num,den)

% We could also express den=[1 -5 6 0]

r = 1.6667 1.5000 -3.1667 p = 3 2 0 k = [] Hence,

$$F[z] = -3.1667 + \frac{1.5z}{z-2} + \frac{1.6667z}{z-3} \quad \bigodot$$

 \triangle Exercise E11.2

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Find the inverse z-transform of the following functions:

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(a)
$$\frac{z(2z-1)}{(z-1)(z+0.5)}$$
 (b) $\frac{1}{(z-1)(z+0.5)}$
(c) $\frac{9}{(z+2)(z-0.5)^2}$ (d) $\frac{5z(z-1)}{z^2-1.6z+0.8}$
Answer: (a) $\left[\frac{2}{3} + \frac{4}{3}(-0.5)^k\right]u[k]$ (b) $-2\delta[k] + \left[\frac{2}{3} + \frac{4}{3}(-0.5)^k\right]u[k]$
(c) $18\delta[k] - [0.72(-2)^k + 17.28(0.5)^k - 14.4k(0.5)^k]u[k]$
(d) $\frac{5\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\right)^k \cos(0.464k + 0.464)u[k]$. Hint: $\sqrt{0.8} = \frac{2}{\sqrt{5}}$. \bigtriangledown

Inverse Transform by Expansion of $\mathbf{F}[\mathbf{z}]$ in Power Series of \mathbf{z}^{-1}

By definition

$$F[z] = \sum_{k=0}^{\infty} f[k] z^{-k}$$

= $f[0] + \frac{f[1]}{z} + \frac{f[2]}{z^2} + \frac{f[3]}{z^3} + \cdots$
= $f[0] z^0 + f[1] z^{-1} + f[2] z^{-2} + f[3] z^{-3} + \cdots$

This result is a power series in z^{-1} . Therefore, if we can expand F[z] into a power series in z^{-1} , the coefficients of this power series can be identified as f[0], f[1], f[2], f[2

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 $f[3], \dots,$ and so on. A rational F[z] can be expanded into a power series of z^{-1} by dividing its numerator by the denominator. Consider, for example,

$$F[z] = \frac{z^2(7z - 2)}{(z - 0.2)(z - 0.5)(z - 1)}$$
$$= \frac{7z^3 - 2z^2}{z^3 - 1.7z^2 + 0.8z - 0.1}$$

To obtain a series expansion in powers of z^{-1} , we divide the numerator by the denominator as follows:

$$\frac{7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots}{7z^3 - 2z^2}$$

$$\frac{7z^3 - 11.9z^2 + 5.60z - 0.7}{9.9z^2 - 5.60z + 0.7}$$

$$\frac{9.9z^2 - 16.83z + 7.92 - 0.99z^{-1}}{11.23z - 7.22 + 0.99z^{-1}}$$

$$\frac{11.23z - 19.09 + 8.98z^{-1}}{11.87 - 7.99z^{-1}}$$

Thus

$$F[z] = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots$$

Therefore

 $f[0] = 7, f[1] = 9.9, f[2] = 11.23, f[3] = 11.87, \dots, and so on.$

We give here a simple MATLAB program to find the first N terms of the inverse z-transform.

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• Computer Example C11.2
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Using MATLAB, find the first 10 values (f[0] through f[9]) of the inverse z-transform of F[z] in the above example.

num=[7 -2 0 0]; den=[1 -1.7 0.8 -0.1]; f=dimpulse(num, den, 10) % We could also write den=conv(conv([1 -0.2],[1 -0.5]),[1 -1]) f =

7.0000 9.9000 11.2300 11.8710 12.1867 12.3436 12.4218 12.4609 12.4805 12.4902 •



Although this procedure yields f[k] directly, it does not provide a closed-form solution. For this reason, it is not very useful unless we want to know only the first few terms of the sequence f[k].

\triangle Exercise E11.3

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Using long division to find the power series in z^{-1} , show that the inverse z-transform of z/(z-0.5) is $(0.5)^k u[k]$ or $(2)^{-k} u[k]$. \bigtriangledown

Relationship Between h[k] and H[z]

For an LTID system, if h[k] is its unit impulse response, then in Eq. (9.57b) we defined H[z], the system transfer function, as

$$H[z] = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

For causal systems, the limits on the sum are from k = 0 to ∞ . This equation shows that the transfer function H[z] is the z-transform of the impulse response h[k] of an LTID system; that is

$$h[k] \Longleftrightarrow H[z] \tag{11.14}$$

This important result relates the impulse response h[k], which is a time-domain specification of a system, to H[z], which is a frequency-domain specification of a system. The result is parallel to that for LTIC systems.

 \triangle Exercise E11.4

Redo Exercise E9.5 by taking the inverse z-transform of H[z]. \bigtriangledown

11.2 Some properties of the Z-Transform

The z-transform properties are useful in the derivation of z-transforms of many functions and also in the solution of linear difference equations with constant coefficients. Here we consider a few important properties of the z-transform.

Right Shift (Delay)

If

$$f[k]u[k] \iff F[z]$$

then

 $f[k-1]u[k-1] \iff \frac{1}{z}F[z]$ (11.15a)

and

$$f[k-m]u[k-m] \iff \frac{1}{z^m}F[z]$$
 (11.15b)

and

$$f[k-1]u[k] \iff \frac{1}{z}F[z] + f[-1]$$
(11.16a)

11.2 Some Properties of z-Transform

Repeated application of this property yields

$$f[k-2]u[k] \iff \frac{1}{z} \left[\frac{1}{z} F[z] + f[-1] \right] + f[-2]$$
$$= \frac{1}{z^2} F[z] + \frac{1}{z} f[-1] + f[-2]$$
(11.16b)

and

$$f[k-m]u[k] \Longleftrightarrow z^{-m}F[z] + z^{-m}\sum_{k=1}^{m} f[-k]z^k$$
(11.16c)

Proof:

$$\mathcal{Z}\left\{f[k-m]u[k-m]\right\} = \sum_{k=0}^{\infty} f[k-m]u[k-m]z^{-k}$$

Recall that f[k-m]u[k-m] = 0 for k < m, so that the limits on the summation on the right-hand side can be taken from k = m to ∞ . Therefore

$$\mathcal{Z}\left\{f[k-m]u[k-m]\right\} = \sum_{k=m}^{\infty} f[k-m]z^{-k}$$
$$= \sum_{r=0}^{\infty} f[r]z^{-(r+m)}$$
$$= \frac{1}{z^m} \sum_{r=0}^{\infty} f[r]z^{-r}$$
$$= \frac{1}{z^m} F[z]$$

To prove Eq. (11.16c), we have

$$\begin{aligned} \mathcal{Z}\left\{f[k-m]u[k]\right\} &= \sum_{k=0}^{\infty} f[k-m]z^{-k} = \sum_{r=-m}^{\infty} f[r]z^{-(r+m)} \\ &= z^{-m} \left[\sum_{r=-m}^{-1} f[r]z^{-r} + \sum_{r=0}^{\infty} f[r]z^{-r}\right] \\ &= z^{-m} \sum_{k=1}^{m} f[-k]z^{k} + z^{-m}F[z] \end{aligned}$$

Left Shift (Advance)

If

$$f[k]u[k] \iff F[z]$$

then

$$f[k+1]u[k] \iff zF[z] - zf[0] \tag{11.17a}$$

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Repeated application of this property yields

$$f[k+2]u[k] \iff z \{ z (F[z] - zf[0]) - f[1] \}$$

= $z^2 F[z] - z^2 f[0] - zf[1]$ (11.17b)

and

$$f[k+m]u[k] \iff z^m F[z] - z^m \sum_{k=0}^{m-1} f[k] z^{-k}$$
 (11.17c)

Proof: By definition

$$\begin{aligned} \mathcal{Z}\left\{f[k+m]u[k]\right\} &= \sum_{k=0}^{\infty} f[k+m]z^{-k} \\ &= \sum_{r=m}^{\infty} f[r]z^{-(r-m)} \\ &= z^m \sum_{r=m}^{\infty} f[r]z^{-r} \\ &= z^m \left[\sum_{r=0}^{\infty} f[r]z^{-r} - \sum_{r=0}^{m-1} f[r]z^{-r}\right] \\ &= z^m F[z] - z^m \sum_{r=0}^{m-1} f[r]z^{-r} \end{aligned}$$



Fig. 11.4 Signal for Example 11.4.

Example 11.4

Find the z-transform of the signal f[k] depicted in Fig. 11.4. The signal f[k] can be expressed as a product of k and a gate pulse u[k] - u[k-6]. Therefore

$$f[k] = k \{u[k] - u[k - 6]\}$$

= ku[k] - ku[k - 6]

11.2 Some Properties of z-Transform

We cannot find the z-transform of ku[k-6] directly by using the right-shift property [Eq. (11.15b)]. So we rearrange it in terms of (k-6)u[k-6] as follows:

$$f[k] = ku[k] - [(k-6)u[k-6] + 6u[k-6]]$$

We can now find the z-transform of the bracketed term by using the right-shift property [Eq. (11.15b)]. Because $u[k] \iff \frac{z}{z-1}$

$$u[k-6] \Longleftrightarrow \frac{1}{z^6} \frac{z}{z-1} = \frac{1}{z^5(z-1)}$$

Also, because $ku[k] \iff \frac{z}{(z-1)^2}$

$$(k-6)u[k-6] \iff \frac{1}{z^6} \frac{z}{(z-1)^2} = \frac{1}{z^5(z-1)^2}$$

Therefore

$$F[z] = \frac{z}{(z-1)^2} - \frac{1}{z^5(z-1)^2} - \frac{6}{z^5(z-1)}$$
$$= \frac{z^6 - 6z + 5}{z^5(z-1)^2} \quad \blacksquare$$

\triangle Exercise E11.5

Using only the fact that $u[k] \iff \frac{z}{z-1}$ and the right-shift property [Eq. (11.15)], find the z-transforms of the signals in Figs. 11.2 and 11.3. The answers are given in Example 11.2d and Exercise E11.1a. \bigtriangledown

Convolution

The time convolution property and the frequency convolution property state that if

$$f_1[k] \iff F_1[z] \text{ and } f_2[k] \iff F_2[z],$$

then (time convolution)

$$f_1[k] * f_2[k] \iff F_1[z]F_2[z] \tag{11.18}$$

and (frequency convolution)

$$f_1[k]f_2[k] \iff \frac{1}{2\pi j} \oint F_1[u]F_2\left[\frac{z}{u}\right] u^{-1} du \qquad (11.19)$$

Proof: These properties apply to causal as well as noncausal sequences. For this reason, we shall prove them for the more general case of noncausal sequences, where the convolution sum ranges from $-\infty$ to ∞ . To prove the time convolution, we have

$$\mathcal{Z}\left\{f_1[k] * f_2[k]\right\} = \mathcal{Z}\left[\sum_{m=-\infty}^{\infty} f_1[m]f_2[k-m]\right]$$
$$= \sum_{k=-\infty}^{\infty} z^{-k} \sum_{m=-\infty}^{\infty} f_1[m]f_2[k-m]$$



Interchanging the order of summation,

$$\mathcal{Z}[f_1[k] * f_2[k]] = \sum_{m=-\infty}^{\infty} f_1[m] \sum_{k=-\infty}^{\infty} f_2[k-m] z^{-k}$$
$$= \sum_{m=-\infty}^{\infty} f_1[m] \sum_{r=-\infty}^{\infty} f_2[r] z^{-(r+m)}$$
$$= \sum_{m=-\infty}^{\infty} f_1[m] z^{-m} \sum_{r=-\infty}^{\infty} f_2[r] z^{-r}$$
$$= F_1[z] F_2[z]$$

To prove the frequency convolution, we start with

$$Z \{ f_1[k] f_2[k] \} = \sum_{k=-\infty}^{\infty} f_1[k] f_2[k] z^{-k}$$
$$= \frac{1}{2\pi j} \sum_{k=-\infty}^{\infty} f_2[k] z^{-k} \oint F_1[u] u^{k-1} du$$

Interchanging the order of summation and integration

$$\begin{aligned} \mathcal{Z}\left[f_1[k]f_2[k]\right] &= \frac{1}{2\pi j} \oint F_1[u] \left[\sum_{k=-\infty}^{\infty} f_2[k] \left(\frac{z}{u}\right)^{-k}\right] u^{-1} du \\ &= \frac{1}{2\pi j} \oint F_1[u]F_2\left[\frac{z}{u}\right] u^{-1} du \end{aligned}$$

LTID System Response

It is interesting to apply the time convolution property to the LTID inputoutput equation y[k] = f[k] * h[k]. In Eq. (11.14), we have shown that $h[k] \iff H[z]$. Hence, according to Eq. (11.18), it follows that

$$Y[z] = F[z]H[z] \tag{11.20}$$

Earlier in the chapter, we derived this important result using informal arguments.

Multiplication by γ^k

If

$$f[k]u[k] \iff F[z]$$

then

$$\gamma^k f[k]u[k] \iff F\left[\frac{z}{\gamma}\right]$$
 (11.21)

Proof:

$$\mathcal{Z}\{\gamma^k f[k]u[k]\} = \sum_{k=0}^{\infty} \gamma^k f[k] z^{-k} = \sum_{k=0}^{\infty} f[k] \left(\frac{z}{\gamma}\right)^{-k} = F\left[\frac{z}{\gamma}\right]$$

11.3 Z-Transform Solution of Linear Difference Equations

\triangle Exercise E11.6

Using Eq. (11.21), derive Pairs 7 and 8 in Table 11.1 from Pairs 2 and 3, respectively. \bigtriangledown

Multiplication by k (Scaling in the z-Domain

If

 $f[k]u[k] \Longleftrightarrow F[z]$

then

$$kf[k]u[k] \iff -z\frac{d}{dz}F[z]$$
 (11.22)

Proof:

$$\begin{aligned} -z\frac{d}{dz}F[z] &= -z\frac{d}{dz}\sum_{k=0}^{\infty}f[k]z^{-k} = -z\sum_{k=0}^{\infty}-kf[k]z^{-k-1} \\ &= \sum_{k=0}^{\infty}kf[k]z^{-k} = \mathcal{Z}\left\{kf[k]u[k]\right\}\end{aligned}$$

 \triangle Exercise E11.7

Using Eq. (11.22), derive Pairs 3 and 4 in Table 11.1 from Pair 2. Similarly, derive Pairs 8 and 9 from Pair 7. \bigtriangledown

Initial and Final Value

For a causal f[k],

$$f[0] = \lim_{z \to \infty} F[z] \tag{11.23a}$$

This result follows immediately from Eq. (11.9)

We can also show that if (z-1)F(z) has no poles outside the unit circle, then

$$\lim_{N \to \infty} f(N) = \lim_{z \to 1} (z - 1) F(z)$$
 (11.23b)

11.3 Z-Transform Solution of Linear Difference Equations

The time-shifting (left- or right-shift) property has set the stage for solving linear difference equations with constant coefficients. As in the case of the Laplace transform with differential equations, the z-transform converts difference equations into algebraic equations which are readily solved to find the solution in the zdomain. Taking the inverse z-transform of the z-domain solution yields the desired time-domain solution. The following examples demonstrate the procedure.

Example 11.5 Solve

$$y[k+2] - 5y[k+1] + 6y[k] = 3f[k+1] + 5f[k]$$
(11.24)

if the initial conditions are $y[-1] = \frac{11}{6}$, $y[-2] = \frac{37}{36}$, and the input $f[k] = (2)^{-k}u[k]$.



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Z- Transform Operations					
Operation	f[k]	F[z]			
Addition	$f_1[k] + f_2[k]$	$F_1[z] + F_2[z]$			
Scalar multiplication	af[k]	aF[z]			
Right-shift	f[k-m]u[k-m]	$rac{1}{z^m}F[z]$			
	f[k-m]u[k]	$\frac{1}{z^m}F[z] + \frac{1}{z^m}\sum_{k=1}^m f[-k]z^k$			
	f[k-1]u[k]	$\frac{1}{z}F[z] + f[-1]$			
	f[k-2]u[k]	$\frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2]$			
	f[k-3]u[k]	$\frac{1}{z^3}F[z] + \frac{1}{z^2}f[-1] + \frac{1}{z}f[-2] + f[-3]$			
Left-shift	f[k+m]u[k]	$z^m F[z] - z^m \sum_{k=0}^{m-1} f[k] z^{-k}$			
	f[k+1]u[k]	zF[z]-zf[0]			
	f[k+2]u[k]	$z^2 F[z] - z^2 f[0] - z f[1]$			
	f[k+3]u[k]	$z^{3}F[z] - z^{3}f[0] - z^{2}f[1] - zf[2]$			
Multiplication by γ^k	$\gamma^k f[k]u[k]$	$F\left[rac{z}{\gamma} ight]$			
Multiplication by k	kf[k]u[k]	$-zrac{d}{dz}F[z]$			
Time Convolution	$f_1[k]\ast f_2[k]$	$F_1[z]F_2[z]$			
Frequency Convolution	$f_1[k]f_2[k]$	$\frac{1}{2\pi j}\oint F_1[u]F_2\left[\frac{z}{u}\right]u^{-1}du$			
Initial value	f[0]	$\lim_{z\to\infty} F[z]$			
Final value	$\lim_{N\to\infty} f[N]$	$\lim_{z\to 1} (z-1)F[z]$ poles of			
		(z-1)F[z] inside the unit circle.			

Table 11.2

11.3 Z-Transform Solution of Linear Difference Equations

As we shall see, difference equations can be solved by using the right-shift or the leftshift property. Because the difference equation (11.24) is in advance-operator form, the use of the left-shift property in Eqs. (11.17a) and (11.17b) may seem appropriate for its solution. Unfortunately, as seen from Eqs. (11.17a) and (11.17b), these properties require a knowledge of auxiliary conditions $y[0], y[1], \dots, y[n-1]$ rather than of the initial conditions $y[-1], y[-2], \dots, y[-n]$, which are generally given. This difficulty can be overcome by expressing the difference equation (11.24) in delay operator form (obtained by replacing k with k-2) and then using the right-shift property.[†] Equation (11.24) in delay operator form is

$$y[k] - 5y[k-1] + 6y[k-2] = 3f[k-1] + 5f[k-2]$$
(11.25)

We now use the right-shift property to take the z-transform of this equation. But before proceeding, we must be clear about the meaning of a term like y[k-1]. Does it mean y[k-1]u[k-1] or y[k-1]u[k]? The answer becomes clear when we recognize that the use of the unilateral transform implies that we are considering the situation for $k \ge 0$, and that every signal in Eq. (11.25) must be counted from k = 0. Therefore, the term y[k-j] means y[k-j]u[k]. Remember also that although we are considering the situation for $k \ge 0$, y[k] is present even before k = 0 (in the form of initial conditions). Now

$$y[k]u[k] \iff Y[z]$$

$$y[k-1]u[k] \iff \frac{1}{z}Y[z] + y[-1] = \frac{1}{z}Y[z] + \frac{11}{6}$$

$$y[k-2]u[k] \iff \frac{1}{z^2}Y[z] + \frac{1}{z}y[-1] + y[-2] = \frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}$$

Also

$$f[k] = (2)^{-k}u[k] = (2^{-1})^{k}u[k] = (0.5)^{k}u[k] \iff \frac{z}{z - 0.5}$$

$$\begin{aligned} f[k-1]u[k] &\iff \frac{1}{z}F[z] + f[-1] = \frac{1}{z}\frac{z}{z-0.5} + 0 = \frac{1}{z-0.5} \\ f[k-2]u[k] &\iff \frac{1}{z^2}F[z] + \frac{1}{z}f[-1] + f[-2] = \frac{1}{z^2}F[z] + 0 + 0 = \frac{1}{z(z-0.5)} \end{aligned}$$

Note that for causal input f[k],

$$f[-1] = f[-2] = \cdots = f[-n] = 0$$

Hence

$$f[k-r]u[k] \iff \frac{1}{z^r}F[z]$$

Taking the z-transform of Eq. (11.25) and substituting the above results, we obtain

$$Y[z] - 5\left[\frac{1}{z}Y[z] + \frac{11}{6}\right] + 6\left[\frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}\right] = \frac{3}{z - 0.5} + \frac{5}{z(z - 0.5)}$$
(11.26a)

or

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y[z] - \left(3 - \frac{11}{z}\right) = \frac{3}{z - 0.5} + \frac{5}{z(z - 0.5)}$$
(11.26b)

 \uparrow Another approach is to find $y[0], y[1], y[2], \dots, y[n-1]$ from $y[-1], y[-2], \dots, y[-n]$ iteratively, as in Sec. 9.1-1, and then apply the left-shift property to Eq. (11.24)



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and

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y[z] = \left(3 - \frac{11}{z}\right) + \frac{3z + 5}{z(z - 0.5)}$$
$$= \frac{3z^2 - 9.5z + 10.5}{z(z - 0.5)}$$

Multiplication of both sides by z^2 yields

$$(z^2 - 5z + 6) Y[z] = \frac{z (3z^2 - 9.5z + 10.5)}{(z - 0.5)}$$

 $Y[z] = \frac{z(3z^2 - 9.5z + 10.5)}{(z - 0.5)(z^2 - 5z + 6)}$

so that

and

$$\frac{Y[z]}{z} = \frac{3z^2 - 9.5z + 10.5}{(z - 0.5)(z - 2)(z - 3)}$$
$$= \frac{(26/15)}{z - 0.5} - \frac{(7/3)}{z - 2} + \frac{(18/5)}{z - 3}$$

 $Y[z] = \frac{26}{15} \left(\frac{z}{z-0.5}\right) - \frac{7}{3} \left(\frac{z}{z-2}\right) + \frac{18}{5} \left(\frac{z}{z-3}\right)$

Therefore

and

$$y[k] = \left[\frac{26}{15}(0.5)^k - \frac{7}{3}(2)^k + \frac{18}{5}(3)^k\right]u[k]$$
(11.28)

(11.27)

This example demonstrates the ease with which linear difference equations with constant coefficients can be solved by z-transform. This method is general; it can be used to solve a single difference equation or a set of simultaneous difference equations of any order as long as the equations are linear with constant coefficients.

Comment

Sometimes auxiliary conditions $y[0], y[1], \dots, y[n-1]$ (instead of initial conditions $y[-1], y[-2], \dots, y[-n]$) are given to solve a difference equation. In this case, the equation can be solved by expressing it in the advance operator form and then using the left-shift property (see Exercise E11.9 below).

\triangle Exercise E11.8

Solve the equation below if the initial conditions are y[-1] = 2, y[-2] = 0, and the input f[k] = u[k]:

$$y[k+2] - \frac{5}{6}y[k+1] + \frac{1}{6}y[k] = 5f[k+1] - f[k]$$

Answer: $y[k] = \left[12 - 15(\frac{1}{2})^k + \frac{14}{3}(\frac{1}{3})^k \right] u[k] \quad \bigtriangledown$

 \triangle Exercise E11.9

Solve the following equation if the auxiliary conditions are y[0] = 1, y[1] = 2, and the input f[k] = u[k]: y[k+2] + 3y[k+1] + 2y[k] = f[k+1] + 3f[k] 11.3 Z-Transform Solution of Linear Difference Equations

Answer: $y[k] = \left[\frac{2}{3} + 2(-1)^k - \frac{5}{3}(-2)^k\right] u[k] \quad \bigtriangledown$

Zero-Input and Zero-State Components

In Example 11.5 we found the total solution of the difference equation. It is relatively easy to separate the solution into zero-input and zero-state components. All we have to do is to separate the response into terms arising from the input and terms arising from initial conditions. We can separate the response in Eq. (11.26b) as follows:

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right)Y[z] - \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{initial condition terms}} = \underbrace{\frac{3}{z - 0.5} + \frac{5}{z(z - 0.5)}}_{\text{terms arising from input}}$$
(11.29)

Therefore

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y[z] = \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{initial condition terms}} + \underbrace{\frac{(3z+5)}{z(z-0.5)}}_{\text{input terms}}$$

Multiplying both sides by z^2 yields

$$(z^2 - 5z + 6) Y[z] = \underbrace{z(3z - 11)}_{\text{initial condition terms}} + \underbrace{\frac{z(3z + 5)}{z - 0.5}}_{\text{input terms}}$$

and

$$Y[z] = \underbrace{\frac{z(3z-11)}{z^2 - 5z + 6}}_{\text{zero-input response}} + \underbrace{\frac{z(3z+5)}{(z-0.5)(z^2 - 5z + 6)}}_{\text{zero-state response}}$$
(11.30)

We expand both terms on the right-hand side into modified partial fractions to yield

$$Y[z] = \underbrace{\left[5\left(\frac{z}{z-2}\right) - 2\left(\frac{z}{z-3}\right)\right]}_{\text{zero-input}} + \underbrace{\left[\frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{22}{3}\left(\frac{z}{z-2}\right) + \frac{28}{5}\left(\frac{z}{z-3}\right)\right]}_{\text{zero-state}}$$

and

$$y[k] = \left[\underbrace{\frac{5(2)^k - 2(3)^k}{\text{zero-input}} - \underbrace{\frac{22}{3}(2)^k + \frac{28}{5}(3)^k + \frac{26}{15}(0.5)^k}_{\text{zero-state}}}\right] u[k]$$
$$= \left[-\frac{7}{3}(2)^k + \frac{18}{5}(3)^k + \frac{26}{15}(0.5)^k\right] u[k]$$

a conclusion, which agrees with the result in Eq. (11.28).

△ Exercise E11.10 Solve

$$y[k+2] - \frac{5}{6}y[k+1] + \frac{1}{6}y[k] = 5f[k+1] - f[k]$$

if the initial conditions are y[-1] = 2, y[-2] = 0, and the input f[k] = u[k]. Separate the response into zero-input and zero-state components.

Answer:

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$$\begin{split} y[k] &= \left\{ \underbrace{[3(\frac{1}{2})^k - \frac{4}{3}(\frac{1}{3})^k]}_{\text{zero-input}} + \underbrace{[12 - 18(\frac{1}{2})^k + 6(\frac{1}{3})^k]}_{\text{zero-state}} \right\} u[k] \\ &= \begin{bmatrix} 12 - 15(\frac{1}{2})^k + \frac{14}{3}(\frac{1}{3})^k \end{bmatrix} u[k] \quad \bigtriangledown$$

11.3-1 Zero-State Response of LTID Systems: The Transfer Function

Consider an nth-order LTID system specified by the difference equation

$$Q[E]y[k] = P[E]f[k]$$
(11.31a)

or

$$E^{n} + a_{n-1}E^{n-1} + \dots + a_{1}E + a_{0})y[k] = (b_{n}E^{n} + b_{n-1}E^{n-1} + \dots + b_{1}E + b_{0})f[k] \quad (11.31b)$$

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 $y[k+n] + a_{n-1}y[k+n-1] + \dots + a_1y[k+1] + a_0y[k]$ = $b_nf[k+n] + \dots + b_1f[k+1] + b_0f[k]$ (11.31c)

We now derive the general expression for the zero-state response; that is, the system response to input f[k] when all the initial conditions $y[-1] = y[-2] = \cdots = y[-n] = 0$ (zero state). The input f[k] is assumed to be causal so that $f[-1] = f[-2] = \cdots = f[-n] = 0$.

Equation (11.31c) can be expressed in the delay operator form as

 $y[k] + a_{n-1}y[k-1] + \dots + a_0y[k-n]$ = $b_nf[k] + b_{n-1}f[k-1] + \dots + b_0f[k-n]$ (11.31d)

Because y[-r] = f[-r] = 0 for r = 1, 2, ..., n

$$y[k-m]u[k] \iff \frac{1}{z^m} Y[z]$$
$$f[k-m]u[k] \iff \frac{1}{z^m} F[z] \qquad m = 1, 2, \dots, n$$

Now the z-transform of Eq. (11.31d) is given by

$$\left(1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_0}{z^n}\right)Y[z] = \left(b_n + \frac{b_{n-1}}{z} + \frac{b_{n-2}}{z^2} + \dots + \frac{b_0}{z^n}\right)F[z]$$

Multiplication of both sides by z^n yields

$$(z^{n} + a_{n-1}z^{n-1} + \dots + a_{1}z + a_{0})Y[z]$$

= $(b_{n}z^{n} + b_{n-1}z^{n-1} + \dots + b_{1}z + b_{0})F[z]$



11.3 Z-Transform Solution of Linear Difference Equations



Fig. 11.5 The transformed representation of an LTID system.

Therefore

$$Y[z] = \left(\frac{b_n z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0}\right) F[z]$$
(11.32)

$$=\frac{P[z]}{Q[z]}F[z] \tag{11.33}$$

We have shown in Eq. (11.20) that Y[z] = F[z]H[z]. Hence, it follows that

$$H[z] = \frac{P[z]}{Q[z]} = \frac{b_n z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0}$$
(11.34)

As in the case of LTIC systems, this result leads to an alternative definition of the LTID system transfer function as the ratio of Y[z] to F[z] (assuming all initial conditions zero).

$$H[z] \equiv \frac{Y[z]}{F[z]} = \frac{\mathcal{Z}[\text{zero-state response}]}{\mathcal{Z}[\text{input}]}$$
(11.35)

Because Y[z], the z-transform of the zero-state response y[k], is the product of F[z] and H[z], we can represent an LTID system in the frequency domain by a block diagram, as illustrated in Fig. 11.5. Just as in continuous-time systems, we can represent discrete-time systems in the transformed manner by representing all signals by their z-transforms and all system components (or elements) by their transfer functions.

Observe that the denominator of H[z] is Q[z], the characteristic polynomial of the system. Therefore the poles of H[z] are the characteristic roots of the system. Consequently, the system stability criterion can be stated in terms of the poles of the transfer function of an LTID system as follows:

- 1. An LTID system is asymptotically stable if and only if all the poles of its transfer function H[z] lie inside a unit circle (centered at the origin) in the complex plane. The poles may be repeated or unrepeated.
- 2. An LTID system is unstable if and only if either one or both of the following conditions exist: (i) at least one pole of H[z] is outside the unit circle; (ii) there are repeated poles of H[z] on the unit circle.
- 3. An LTID system is marginally stable if and only if there are no poles of H[z] outside the unit circle, and there are some unrepeated poles on the unit circle.



f[k] u[k]	1	f [k-1] u [k-1]
F[z]	z	$Y[z] = \frac{1}{z} F[z]$

Fig. 11.6 Ideal unit delay and its transfer function.

Example 11.6: The Transfer Function of a Unit Delay Show that the transfer function of a unit delay is 1/z.

If the input to the unit delay is f[k]u[k], then its output (Fig. 11.6) is given by

y[k] = f[k-1]u[k-1]

The z-transform of this equation yields [see Eq. (11.15a)]

$$Y[z] = \frac{1}{z}F[z]$$
$$= H[z]F[z]$$

It follows that the transfer function of the unit delay is

$$H[z] = \frac{1}{z} \tag{11.36}$$

Example 11.7

Find the response y[k] of an LTID system described by the difference equation

y[k+2] + y[k+1] + 0.16y[k] = f[k+1] + 0.32f[k]

or

$$(E^{2} + E + 0.16)y[k] = (E + 0.32)f[k]$$

for the input $f[k] = (-2)^{-k} u[k]$ and with all the initial conditions zero (system in zero state initially).

From the difference equation we find

$$H[z] = \frac{P[z]}{Q[z]} = \frac{z + 0.32}{z^2 + z + 0.16}$$

For the input $f[k] = (-2)^{-k} u[k] = [(-2)^{-1}]^k u(k) = (-0.5)^k u[k]$
$$F[z] = \frac{z}{z + 0.5}$$

$$Y[z] = F[z]H[z] = \frac{z(z+0.32)}{(z^2+z+0.16)(z+0.5)}$$

Therefore

and

$$\frac{Y[z]}{z} = \frac{(z+0.32)}{(z^2+z+0.16)(z+0.5)} = \frac{(z+0.32)}{(z+0.2)(z+0.8)(z+0.5)}$$
$$= \frac{2/3}{z+0.2} - \frac{8/3}{z+0.8} + \frac{2}{z+0.5}$$
(11.37)

11.4 System Realization

so that

$$Y[z] = \frac{2}{3} \left(\frac{z}{z+0.2}\right) - \frac{8}{3} \left(\frac{z}{z+0.8}\right) + 2 \left(\frac{z}{z+0.5}\right)$$
(11.38)

and

$$3(z+0.2)$$
 $3(z+0.8)$ $(z+0.5)$

$$y[k] = \left[\frac{2}{3}(-0.2)^k - \frac{8}{3}(-0.8)^k + 2(-0.5)^k\right]u[k]$$

 \odot Computer Example C11.3

Solve Example 11.7 using MATLAB. Plot y[k] for $0 \le k \le 10$.

k=0:10; $b = [0 \ 1 \ 0.32];$ $a = [1 \ 1 \ 0.16];$ $f=(-2).^{(-k)};$ y=filter(b,a,f); stem(k,y) xlabel('k');ylabel('y[k]') (\cdot)

Exercise E11.11 \wedge

A discrete-time system is described by the following transfer function:

$$H[z] = \frac{z - 0.5}{(z + 0.5)(z - 1)}$$

(a) Find the system response to input $f[k] = 3^{-(k+1)}u[k]$ if all initial conditions are zero. (b) Write the difference equation relating the output y[k] to input f[k] for this system. Answers: (a) $y[k] = \frac{1}{3} \left[\frac{1}{2} - 0.8(-0.5)^k + 0.3 \left(\frac{1}{3} \right)^k \right] u[k]$ (b) y[k+2] - 0.5y[k+1] - 0.5y[k] = f[k+1] - 0.5f[k] ∇

11.4System Realization

We now discuss ways to realize an nth-order discrete-time system described by a transfer function

$$H[z] = \frac{b_n z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0}$$
(11.39)

This transfer function is identical to the general nth-order continuous-time transfer function H(s) in Eq. (6.70) with s replaced by z. It is reasonable to believe that the realization of H[z] in (11.39) would be identical to that of H(s) with s replaced by z. Fortunately this happens to be the case. In realizations of H(s) the basic element used was an integrator with transfer function 1/s. In realizations of H[z] the basic element is unit delay with transfer function 1/z. Therefore, all the realizations of H(s) studied in Sec. 6.6 are also the realizations of H[z] if we replace integrators by unit delays. To demonstrate this point, consider a realization of a third-order transfer function.

$$H[z] = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$
(11.40)

Figure 11.7 shows Fig. 6.21 with all the integrators (with transfer function 1/s) replaced with unit delays (with transfer function 1/z). We shall now show that this





Fig. 11.7 A canonical realization of H[z].

realization indeed represents H[z] in Eq. (11.40). Let the signal at the output of the third delay be X[z]. Consequently, signals at the inputs of the second and the first delay are zX[z] and $z^2X[z]$. The first summer output $z^3X[z]$ is equal to the sum of the four inputs to that summer. Therefore

$$z^{3}X[z] = -a_{2}z^{2}X[z] - a_{1}zX[z] - a_{0}X[z] + F[z]$$

so that

$$(z^{3} + a_{2}z^{2} + a_{1}z + a_{0})X[z] = F[z]$$
(11.41)

Moreover, Y[z], the output of the second summer, is equal to the sum of four signals to that summer. Therefore

$$Y[z] = (b_3 z^3 + b_2 z^2 + b_1 z + b_0) X[z]$$
(11.42)

From Eqs. (11.41) and (11.42), it follows that

$$\frac{Y[z]}{F[z]} = \frac{b_3 z^3 + b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

This result shows that Fig. 11.7 is indeed a realization of H[z] in Eq. (11.40). Similarly, the cascade and parallel realizations of the continuous-time case are directly applicable to discrete-time systems, with integrators replaced by unit delays. The second canonical realization developed in Appendix 6.1 also applies to discrete-time case with 1/s replaced by 1/z.

11.5 Connection between The Laplace and the Z-Transforms

Example 11.8

Realize the following transfer functions, using only the cascade form for part a and using only the parallel form for part b.

(a)
$$H[z] = \frac{4z+28}{z^2+6z+5}$$
 (b) $H[z] = \frac{7z^2+37z+51}{(z+2)(z+3)^2}$

Identical transfer functions for continuous-time systems are realized in Figs. 6.27 and 6.28.

\triangle Exercise E11.12

Give the canonical realization of the following transfer functions. (a) $\frac{2}{z+5}$ (b) $\frac{z+8}{z+5}$

(c) $\frac{z}{z+5}$ (d) $\frac{2z+3}{z^2+6z+25}$

Answer: See Example 6.18. Replace 1/s by 1/z and make appropriate changes in coefficients. ∇

11.5 Connection between the Laplace and the Z-Transform

We now show that discrete-time systems also can be analyzed using the Laplace transform. In fact, we shall see that the z-transform is the Laplace transform in disguise and that discrete-time systems can be analyzed as if they were continuoustime systems.

So far we have considered the discrete-time signal as a sequence of numbers and not as an electrical signal (voltage or current). Similarly, we have considered a discrete-time system as a mechanism that processes a sequence of numbers (input) to yield another sequence of numbers (output). The system was built by using delays (along with adders and multipliers) that delay sequences of numbers, not electrical signals (voltages or currents). A digital computer is a perfect example: every signal is a sequence of numbers, and the processing involves delaying sequences of numbers (along with addition and multiplication).

Consider a discrete-time system with transfer function H[z] and an input f[k], as shown in Fig. 11.8a. We can think of (or generate, for that matter) a corresponding continuous-time signal f(t) consisting of impulses spaced T seconds apart. Let the kth impulse of strength be f[k] as depicted in Fig. 11.8b. Thus

$$\tilde{f}(t) = \sum_{k=0}^{\infty} f[k]\delta(t - kT)$$
 (11.43)

Figure 11.8 shows f[k] and corresponding $\tilde{f}(t)$. Let us now consider a system identical in structure to the discrete-time system with transfer function H[z], except that the delays in H[z] are replaced by elements that delay continuous-time signals (such as voltages or currents). If a continuous-time impulse $\delta(t)$ is applied to such a delay of T seconds, the output will be $\delta(t-T)$. The continuous-time transfer function of such a delay is e^{-sT} [see Eq. (6.54)]. Hence, the delay elements with transfer function 1/z in the realization of H[z] will be replaced by the delay elements with transfer function e^{-sT} in the realization of the corresponding $\tilde{H}(s)$. This step is the same as z being replaced by e^{sT} . Therefore, the transfer function of this system is H[z] with z replaced by e^{sT} . Thus $\tilde{H}(s) = H[e^{sT}]$. Now whatever operations are performed by the discrete-time system H[z] on f[k] (Fig. 11.8a) are also performed by the corresponding continuous-time system $H[e^{sT}]$ on the impulse sequence $\tilde{f}(t)$



Fig. 11.8 Connection between the Laplace and z-transform.

(Fig. 11.8b). The delaying of a sequence in H[z] would amount to the delaying of an impulse train in $H[e^{sT}]$. The case of adding and multiplying operations is similar. In other words, one-to-one correspondence of the two systems is preserved in every aspect. Therefore, if y[k] is the output of the discrete-time system in Fig. 11.8a, then $\tilde{y}(t)$, the output of the continuous-time system in Fig. 11.8b, would be a sequence of impulses whose kth impulse strength is y[k]. Thus

$$\tilde{y}(t) = \sum_{k=0}^{\infty} y[k]\delta(t-kT)$$
(11.44)

The system in Fig. 11.8b, being a continuous-time system, can be analyzed by using the Laplace transform. If

$$\tilde{f}(t) \iff \tilde{F}(s) \quad \text{and} \quad \tilde{y}(t) \iff \tilde{Y}(s)$$

then

$$\tilde{Y}(s) = H[e^{sT}]\tilde{F}(s) \tag{11.45}$$

11.6 Sampled-Data (Hybrid) Systems

Also

$$\tilde{F}(s) = \mathcal{L}\left[\sum_{k=0}^{\infty} f[k]\delta(t-kT)\right]$$

Now, because the Laplace transform of $\delta(t-kT)$ is e^{-skT}

$$\tilde{F}(s) = \sum_{k=0}^{\infty} f[k] e^{-skT}$$
(11.46)

Similarly

or

where

$$\tilde{Y}(s) = \sum_{k=0}^{\infty} y[k] e^{-skT}$$
 (11.47)

Substitution of Eqs. (11.46) and (11.47) in Eq. (11.45) yields

$$\sum_{k=0}^{\infty} y[k]e^{-skT} = H[e^{sT}] \left[\sum_{k=0}^{\infty} f[k]e^{-skT}\right]$$

By introducing a new variable $z = e^{sT}$, this equation can be expressed as

$$\sum_{k=0}^{\infty} y[k] z^{-k} = H[z] \sum_{k=0}^{\infty} f[k] z^{-k}$$

$$Y[z] = H[z]F[z]$$

$$F[z] = \sum_{k=0}^{\infty} f[k] z^{-k}$$
 and $Y[z] = \sum_{k=0}^{\infty} y[k] z^{-k}$

It is clear from this discussion that the z-transform can be considered as a Laplace transform with a change of variable $z = e^{sT}$ or $s = (1/T) \ln z$. On the other hand, we may consider the z-transform as an independent transform in its own right. Note that the transformation $z = e^{sT}$ transforms the imaginary axis in the s-plane $(s = j\omega)$ into a unit circle in the z-plane $(z = e^{sT} = e^{j\omega T}, \text{ or } |z| = 1)$. The LHP and RHP in the s-plane map into the inside and the outside, respectively, of the unit circle in the z-plane.

11.6 Sampled-data (Hybrid) Systems

Sampled-data systems are hybrid systems consisting of discrete-time as well as continuous-time subsystems. Consider, for example, a fire control system. In this case, the problem is to search and track a moving target and fire a projectile for a direct hit. The data obtained from the search and tracking radar is discretetime data because of a scanning operation, which results in sampling of azimuth, elevation, and the target velocity. This data is now fed to a digital (discrete-time) processor, which performs extensive computations. The computer output is then fed to a continuous-time plant, such as a gun mount, which accordingly positions

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Fig. 11.9 Typical sampled-data systems.

itself at a certain position and fires. Another example is attitude-control problem in a spacecraft, where the information about the actual spacecraft attitude is fed back to a digital processor, which generates corrective input to be applied to the spacecraft, which is a continuous-time system. In automatic periodic quality check in production line, the discrete-data obtained from the periodic check, after some digital processing, generates the corrective input to be applied to a continuous-time plant. In complex control systems, use of digital processor as a controller or a compensator for continuous-time plants is growing rapidly.

In time-sharing systems, where, for economic reasons, certain facilities are shared by several systems, the signals are, by nature, discrete-time or sampled. In regulator type control systems, where an output variable must be maintained at a constant value, the external disturbance and plant parameters variations are usually so slow that continuous monitoring (or feedback) is unnecessary. It is adequate to sample the output periodically and then feed back this discrete-data. In such cases, feedback transducers, data-processing facilities and possibly long and expensive feedback communication facilities can be shared among several control systems.

Figure 11.9 shows some typical sampled-data systems. Figure 11.9a contains a digital processor, whereas in Fig. 11.9b, the sampled signal is directly applied to D/A converter (the hold circuit) without further digital processing. Figure 11.9c shows a practical system, where the input signal itself is a discrete-time signal f[k], and the sampler is in the feedback path. This system is equivalent to that in Fig. 11.9b. How do we analyze such hybrid systems, where continuous-time and discrete-time signals intermingle? An effective strategy in such a situation is to

11.6 Sampled-Data (Hybrid) Systems

relate the samples of the output to those of the input. But, this procedure yields information about the output only at sampling instants. We can overcome this difficulty by taking the samples at instants in between samples using the modified z-transform as explained later.

In sampled-data systems, the discrete-time signals are often obtained as a result of sampling continuous-time signals. These samples are narrow pulses, which may be considered as impulses, provided the pulse width is small compared to the system time constant. Thus, in the following discussion, a discrete-time signal, when it appears in conjunction with a continuous-time system, is a sequence of impulses rather than a sequence of numbers. Hence, a discrete-time signal f[k] can also be considered as continuous-time signal f(t), where

$$f(t) = \sum_{k} f[k]\delta(t - kT)$$

Observe an interesting fact: in this representation a discrete-time unit impulse $\delta[k]$ is the continuous-time unit impulse $\delta(t)$. Thus, at the input of a discrete-time processor, a discrete-time signal f[k] is just a sequence of numbers. But at the input of a continuous-time system, f[k] is a sequence of impulses. There are appropriate converters at the interface of discrete-time and continuous-time systems to carry out signal conversion to appropriate forms.

To begin with, consider a basic continuous-time system (Fig. 11.10a) with transfer function H(s). The input f(t) is sampled and the sampled signal f[k] is applied to the input of H(s). Although y(t), the output of this system, is continuous, we shall endeavor to find the values of y(t) only at the discrete instants t = kT. Such an analysis is relatively simple using the method of z-transform. For this purpose, we consider as if the output is sampled by an hypothetical sampler shown dotted in Fig. 11.10a. Now, we shall relate the input samples f[k] and the output samples y[k]. Let h[k] be the unit impulse response relating the output samples to the input samples. In other words, y[k] = h[k] * f[k]. Recall also that an unit impulse $\delta[k]$ is $\delta(t)$ when considered in conjunction with a continuous-time system. Hence, h[k], the unit impulse response is the sampled version of the system's unit impulse response h(t). Thus,

h[k] = h(kT)

where T is the sampling interval. For instance, if $H(s) = \frac{1}{s-\lambda}$, then $h(t) = e^{\lambda t}$ and

$$h[k] = e^{\lambda kT}$$

Therefore, the equivalent discrete-time transfer function H[z] of this system is given by

$$\begin{aligned} \ddot{H}[z] &= \mathcal{Z}\{h[k]\} \\ &= \mathcal{Z}[e^{\lambda kT}] \\ &= \frac{z}{z - e^{\lambda T}} \end{aligned} \tag{11.49}$$









Fig. 11.10 Computing the output in hybrid or sampled-data systems.

Thus, H[z] is the discrete-time transfer function of $H(s) = \frac{1}{s+\lambda}$ that relates y[k] (the output samples) to the discrete-time input f[k].

If we have two systems with transfer functions G(s) and H(s) in cascade (Fig. 11.10b), the equivalent transfer $T[z] \neq G[z]H[z]$, but is GH[z], where G[z], H[z] and GH[z] correspond to discrete-time transfer functions of G(s), H(s) and G(s)H(s), respectively. For instance, if

$$G(s) = \frac{1}{s+2}$$
 and $H(s) = \frac{1}{s}$

Then, according to Eq. (11.49)

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$$G[z] = rac{z}{z - e^{-2T}}$$
 and $H[z] = rac{z}{z - 1}$

However, the continuous-time system transfer function is G(s)H(s), where

$$G(s)H(s) = \frac{1}{s(s+2)} = \frac{1}{2} \left[\frac{1}{s} - \frac{1}{s+2} \right]$$

And from Eq. (11.49)

[†]Using this procedure, we have listed H(s) and corresponding H[z] in Table 12.1 in Chapter 12. In this Table, H[z] is multiplied with a scaling factor T, which results in $H[z] = \frac{Tz}{z - e^{\lambda T}}$. For the purpose of the sampled data application, the extra factor T should be ignored throughout in Table 12.1.

11.6 Sampled-Data (Hybrid) Systems

$$T[z] = \frac{1}{2} \left(\frac{z}{z-1} - \frac{z}{z-e^{-2T}} \right) \neq G[z]H[z]$$

In this case, we use the notation GH[z] for T[z]. Thus, $GH[z] \neq G[z]H[z]$, but is the discrete-time transfer function which corresponds to G(s)H(s).

For the system in Fig. 11.10c,

$$Y[z] = H[z]X[z] = H[z]G[z]F[z] \quad \text{so that} \quad T[z] = G[z]H[z]$$

For the system in Fig. 11.10d,

W[z] = H[z]Y[z]Y[z] = G[z]E[z]

$$E[z] = F[z] - W[z]$$

Moreover,

Hence

$$Y[z] = \frac{G[z]}{1 + G[z]H[z]}F[z]$$

= G[z] (F[z] - W[z])= G[z](F[z] - H[z]Y[z])

Consequently

$$T[z] = \frac{G[z]}{1 + G[z]H[z]}$$

Example 11.9

Find the output samples y[k] for the sampled-data system illustrated in Fig. 11.11a when the input is a unit step function u(t), the sampling interval T = 0.5 second and

$$G_c[z] = \frac{z}{z-1}$$
 and $G(s) = \frac{1}{s+4}$

This system has a discrete-time controller and a continuous-time plant.[†] To find the transfer function of this system, we observe that

 $Y[z]=G[z]X[z], \qquad X[z]=G_c[z]E[z], \qquad \text{and} \qquad E[z]=F[z]-Y[z]$

Hence

$$Y[z] = G_c[z]G[z](F[z] - Y[z])$$

and the system transfer function T[z] is

$$T[z] = \frac{Y[z]}{F[z]} = \frac{G_c[z]G[z]}{1 + G_c[z]G[z]}$$

[†]The block diagram in Fig. 11.11a does not show the appropriate converters required at the interface of discrete-time and continuous-time systems; these are implied. Thus, the output of the sampler, which consists of impulse sequence, is converted into sequence of numbers to act as the input to the discrete-time controller. Similarly, the output of a discrete-time controller, which is a sequence of numbers, is converted to a sequence of impulses to act as an input to the plant.







For G(s) = 1/(s+4) and T = 0.5, we find, from Eq. (11.49), $G[z] = \frac{z}{z-e^{-2}} = \frac{z}{z-0.1353}$. Also $G_c[z] = z/(z-1)$. Substitution of these expressions in T[z] yields

$$T[z] = \frac{z^2}{(z - 0.394)(z - 0.174)}$$

The output Y[z] is given by Y[z] = T[z]F[z]. For the step input f(t) = u(t), the corresponding sampled signal is u[k] so that F[z] = z/(z-1). Hence,

$$Y[z] = T[z]F[z] = \frac{z^3}{(z-1)(z-0.394)(z-0.174)}$$

and

$$\frac{Y[z]}{z} = \frac{z^2}{(z-1)(z-0.394)(z-0.174)}$$
$$= \frac{1}{z-1} - \frac{0.583}{z-0.394} + \frac{0.083}{z-0.174}$$

11.6 Sampled-Data (Hybrid) Systems

Hence

and

$$Y[z] = \frac{z}{z-1} - \frac{0.583z}{z-0.394} + \frac{0.083z}{z-0.174}$$

 $y[k] = 1 - 0.583(0.394)^k + 0.083(0.174)^k$

This response is depicted in Fig. 11.11.

11.6-1 Response Between Sampling Instants: The Modified Z-Transform

The above analysis yields the output only at sampling instants. We can readily find the response between successive sampling instants by using the modified *z*-transform. This goal can be accomplished by considering the response at another set of sampling instants $t = (k + \mu)T$, where $0 < \mu < 1$.

Consider the system in Fig. 11.10a with $H(s) = 1/(s-\lambda)$. The impulse response is $h(t) = e^{\lambda t}$ and its samples at instants $t = (k + \mu)T$ are

$$h(t, \mu) = e^{\lambda(k+\mu)T} = e^{\lambda\mu T} \left[e^{\lambda kT} \right]$$
(11.50a)

The corresponding z-transfer function is

$$H[z, \mu] = e^{\lambda \mu T} \frac{z}{z - e^{\lambda T}} = \frac{z e^{\lambda \mu T}}{z - e^{\lambda T}}$$
(11.50b)

In this manner, We can prepare a table of modified z-transform. When we use $H[z, \mu]$ instead of H[z] in our analysis, we obtain the response at instants $t = (k + \mu)T$. By using different values of μ in the range 0 to T, we can obtain the complete response y(t).

Example 11.10

Find the output y(t) for all t in Example 11.9.

In Example 11.9, we found the response y[k] only at the sampling instants. To find the output values between sampling instants, we use the modified z-transform. The procedure is the same as before, except that we use modified z-transform corresponding to continuous-time systems and signals. For the system G(s) = 1/s + 4 with T = 0.5, the modified z-transform [Eq. (11.50b) with $\lambda = -4$, and T = 0.5] is

$$H[z, \mu] = e^{\lambda \mu T} \frac{z}{z - e^{-2}} = e^{-2\mu} \frac{z}{z - 0.1353}$$

Moreover to find the modified z-transform corresponding to f(t) = u(t) [$\lambda = 0$ in Eq. (11.50a)], we have $F[z, \mu] = z/(z-1)$. Substitution of these expressions in those found in Example 11.9, we obtain

$$Y[z, \mu] = e^{-2\mu} \left[\frac{z}{z-1} - \frac{0.583z}{z-0.394} + \frac{0.083z}{z-0.174} \right]$$

From Eqs. (11.50), we obtain the inverse (modified) z-transform of this equation as

$$y[(k+\mu)T] = e^{-2\mu} \left[1 - 0.583(0.394)^k + 0.083(-0.174)^k \right] \qquad 0 < \mu < 1$$

The complete response is also shown in Fig. 11.11.

Design of Sampled-Data Systems

As with continuous-time control systems, sampled-data systems are designed to meet certain transient (PO, t_r , t_s , etc.) and steady-state specifications. The design procedure follows along the lines similar to those used for continuous-time systems. We begin with a general second-order system. the relationship between closedloop pole locations and the corresponding transient parameters PO, t_r , t_s ,... are determined. Hence, for a given transient specifications, an acceptable region in the z-plane where the dominant poles of the closed-loop transfer function T[z] should lie is determined. Next, we sketch the root locus for the system. The rules for sketching the root locus are the same as those for continuous-time systems. If the root locus passes through the acceptable region, the transient specifications can be met by simple adjustment of the gain K. If not, we must use a compensator, which will steer the root locus in the acceptable region.

11.7 The Bilateral Z-Transform

Situations involving noncausal signals or systems cannot be handled by the (unilateral) z-transform discussed so far. Such cases can be analyzed by the **bilateral** (or two-sided) z-transform defined by

$$F[z] \equiv \sum_{k=-\infty}^{\infty} f[k] z^{-k}$$

The inverse z-transform is given by

$$f[k] = \frac{1}{2\pi j} \oint F[z] z^{k-1} \, dz$$

These equations define the bilateral z-transform. The unilateral z-transform discussed so far is a special case, where the input signals are restricted to be causal. Restricting signals in this way results in considerable simplification in the region of convergence. Earlier, we showed that

$$\gamma^{k}u[k] \iff \frac{z}{z-\gamma} \qquad |z| > |\gamma|$$
 (11.51)

In contrast, the z-transform of the signal $-\gamma^k u[-(k+1)]$, illustrated in Fig. 11.12a, is

11.7 The Bilateral Z-Transform



Fig. 11.12 $-\gamma^k u[-(k+1)]$ and the region of convergence of its z-transform.

$$\begin{aligned} \mathcal{Z}\left\{-\gamma^{k}u[-(k+1)]\right\} &= \sum_{-\infty}^{-1} -\gamma^{k}z^{-k} = \sum_{-\infty}^{-1} -\left(\frac{\gamma}{z}\right)^{k} \\ &= -\left[\frac{z}{\gamma} + \left(\frac{z}{\gamma}\right)^{2} + \left(\frac{z}{\gamma}\right)^{3} + \cdots\right] \\ &= 1 - \left[1 + \frac{z}{\gamma} + \left(\frac{z}{\gamma}\right)^{2} + \left(\frac{z}{\gamma}\right)^{3} + \cdots\right] \\ &= 1 - \frac{1}{1 - \frac{z}{\gamma}} \qquad \left|\frac{z}{\gamma}\right| < 1 \\ &= \frac{z}{z - \gamma} \qquad |z| < |\gamma| \end{aligned}$$

Therefore

$$\mathcal{Z}\left\{-\gamma^{k}u[-(k+1)]\right\} = \frac{z}{z-\gamma} \qquad |z| < |\gamma| \qquad (11.52)$$

A comparison of Eqs. (11.51) with (11.52) shows that the z-transform of $\gamma^k u[k]$ is identical to that of $-\gamma^k u[-(k+1)]$. The regions of convergence, however, are different. In the former case, F[z] converges for $|z| > |\gamma|$; in the latter, F[z] converges for $|z| < |\gamma|$ (see Fig. 11.12b). Clearly, the inverse transform of F[z] is not unique unless the region of convergence is specified. If we restrict all our signals to be causal, however, this ambiguity does not arise. The inverse transform of $z/(z - \gamma)$ is $\gamma^k u[k]$ even without specifying the region of convergence. Thus, in the unilateral transform, we can ignore the region of convergence in determining the inverse z-transform of F[z].

Example 11.11

Determine the z-transform of

$$f[k] = (0.9)^{k} u[k] + (1.2)^{k} u[-(k+1)]$$

= $f_1[k] + f_2[k]$





Fig. 11.13 Signal f[k] for Example 11.11.

From the results in Eqs. (11.51) and (11.52), we have

$$F_1[z] = \frac{z}{z - 0.9} \qquad |z| > 0.9$$
$$F_2[z] = \frac{-z}{z - 1.2} \qquad |z| < 1.2$$

The common region where both $F_1[z]$ and $F_2[z]$ converge is 0.9 < |z| < 1.2 (Fig. 11.13a). Hence

$$F[z] = F_1[z] + F_2[z]$$

= $\frac{z}{z - 0.9} - \frac{z}{z - 1.2}$
= $\frac{-0.3z}{(z - 0.9)(z - 1.2)}$ 0.9 < $|z| < 1.2$ (11.53)

The sequence f[k] and the region of convergence of F[z] are depicted in Fig. 11.13.

Example 11.12

Find the inverse z-transform of

$$F[z] = \frac{-z(z+0.4)}{(z-0.8)(z-2)}$$

if the region of convergence is (a) |z| > 2 (b) |z| < 0.8 (c) 0.8 < |z| < 2.

(a)
$$\frac{F[z]}{z} = \frac{-(z+0.4)}{(z-0.8)(z-2)}$$
$$= \frac{1}{z-0.8} - \frac{2}{z-2}$$

and

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 $F[z] = \frac{z}{z-0.8} - 2\frac{z}{z-2}$

11.7 The Bilateral Z-Transform



Fig. 11.14 Three possible inverse transforms of F[z] in Example 11.12.

Since the region of convergence is |z| > 2, both terms correspond to causal sequences and

$$f[k] = \left[(0.8)^k - 2(2)^k \right] u[k]$$

This sequence appears in Fig. 11.14a.

(b) In this case, |z| < 0.8, which is less than the magnitudes of both poles. Hence, both terms correspond to anticausal sequences, and

$$f[k] = \left[-(0.8)^k + 2(2)^k \right] u \left[-(k+1) \right]$$

This sequence appears in Fig. 11.14b.

(c) In this case, 0.8 < |z| < 2; the part of F[z] corresponding to the pole at 0.8 is a causal sequence, and the part corresponding to the pole at 2 is an anticausal sequence:

$$f[k] = (0.8)^{k} u[k] + 2(2)^{k} u[-(k+1)]$$

This sequence appears in Fig. 11.14c.

△ Exercise E11.13 Find the inverse z-transform of

$$F[z] = \frac{z}{z^2 + \frac{5}{6}z + \frac{1}{6}} \qquad \qquad \frac{1}{2} > |z| > \frac{1}{3}$$

Answer: $6(-\frac{1}{3})^k u[k] + 6(-\frac{1}{2})^k u[-(k+1)] \quad \bigtriangledown$

Inverse Transform by Expansion of $\mathbf{F}[\mathbf{z}]$ in Power Series of \mathbf{z}

We have

$$F[z] = \sum_{k} f[k] z^{-k}$$

For an anticausal sequence, which exists only for $k \leq -1$, this equation becomes

$$F[z] = f[-1]z + f[-2]z^{2} + f[-3]z^{3} + \cdots$$

We can find the inverse z-transform of F[z] by dividing the numerator polynomial with the denominator polynomial, both in ascending powers of z, to obtain a polynomial in ascending powers of z. Thus, to find the inverse transform of z/(z-0.5)(when the region of convergence is |z| < 0.5), we divide z with -0.5 + z to obtain $-2z - 4z^2 - 8z^3 - \cdots$. Hence, f[-1] = -2, f[-2] = -4, f[-3] = -8 and so on.

11.7-1 Analysis of LTID Systems Using the Bilateral Z-Transform

Because the bilateral z-transform can handle noncausal signals, we can analyze noncausal linear systems using this transform. The zero-state response y[k] is given by

$$y[k] = \mathcal{Z}^{-1} \left\{ F[z]H[z] \right\}$$

provided that F[z]H[z] exists. The region of convergence of F[z]H[z] is the region where both F[z] and H[z] exist, a fact which means that the region is common to the convergence of both F[z] and H[z].

Example 11.13

For a causal system specified by the transfer function

$$H[z] = \frac{z}{z - 0.5}$$

find the zero-state response to input

$$f[k] = (0.8)^{k} u[k] + 2(2)^{k} u[-(k+1)]$$

The z-transform of this signal is found from Example 11.12 (part c) as

$$F[z] = \frac{-z(z+0.4)}{(z-0.8)(z-2)} \qquad 0.8 < |z| < 2$$

Therefore

$$Y[z] = F[z]H[z] = \frac{-z^2(z+0.4)}{(z-0.5)(z-0.8)(z-2)}$$



11.7 The Bilateral Z-Transform

Since the system is causal, the region of convergence of H[z] is |z| > 0.5. The region of convergence of F[z] is 0.8 < |z| < 2. The common region of convergence for F[z] and H[z] is 0.8 < |z| < 2. Therefore

$$Y[z] = \frac{-z^2(z+0.4)}{(z-0.5)(z-0.8)(z-2)} \qquad 0.8 < |z| < 2$$

Expanding Y[z] into modified partial fractions yields

$$Y[z] = -\frac{z}{z-0.5} + \frac{8}{3} \left(\frac{z}{z-0.8}\right) - \frac{8}{3} \left(\frac{z}{z-2}\right) \qquad 0.8 < |z| < 2$$

The poles at 0.5 and 0.8 are enclosed within the ring of convergence and therefore correspond to the causal part, and the pole at 2 is outside the ring of convergence and corresponds to the anticausal part of Y[z]. Therefore

 $y[k] = \left[-(0.5)^k + \frac{8}{3}(0.8)^k \right] u[k] + \frac{8}{3}(2)^k u[-(k+1)]$

Example 11.14

For the system in Example 11.13 find the zero-state response to input

$$f[k] = \underbrace{(0.8)^k u[k]}_{f_1[k]} + \underbrace{(0.6)^k u[-(k+1)]}_{f_2[k]}$$

The z-transforms of the causal and anticausal components $f_1[k]$ and $f_2[k]$ of the output are

$$F_1[z] = rac{z}{z-0.8}$$
 $|z| > 0.8$
 $F_2[z] = rac{-z}{z-0.6}$ $|z| < 0.6$

Observe that a common region of convergence for $F_1[z]$ and $F_2[z]$ does not exist. Therefore F[z] does not exist. In such a case we take advantage of the superposition principle and find $y_1[k]$ and $y_2[k]$, the system responses to $f_1[k]$ and $f_2[k]$, separately. The desired response y[k] is the sum of $y_1[k]$ and $y_2[k]$. Now

$$H[z] = \frac{z}{z - 0.5} \qquad |z| > 0.5$$

$$\begin{split} Y_1[z] &= F_1[z]H[z] = \frac{z^2}{(z-0.5)(z-0.8)} \qquad |z| > 0.8 \\ Y_2[z] &= F_2[z]H[z] = \frac{-z^2}{(z-0.5)(z-0.6)} \qquad 0.5 < |z| < 0.6 \end{split}$$

Expanding $Y_1[z]$ and $Y_2[z]$ into modified partial fractions yields

$$Y_1[z] = -\frac{5}{3} \left(\frac{z}{z-0.5}\right) + \frac{8}{3} \left(\frac{z}{z-0.8}\right) \qquad |z| > 0.8$$
$$Y_2[z] = 5 \left(\frac{z}{z-0.5}\right) - 6 \left(\frac{z}{z-0.6}\right) \qquad 0.5 < |z| < 0.6$$

Therefore

$$y_1[k] = \left[-\frac{5}{3} (0.5)^k + \frac{8}{3} (0.8)^k \right] u[k]$$

$$y_2[k] = 5(0.5)^k u[k] + 6(0.6)^k u[-(k+1)]$$

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and

$$y[k] = y_1[k] + y_2[k]$$

= $\left[\frac{10}{3}(0.5)^k + \frac{8}{3}(0.8)^k\right] u[k] + 6(0.6)^k u[-(k+1)]$

 \triangle Exercise E11.14

For a causal system in Example 11.13, find the zero-state response to input

$$f[k] = \left(\frac{1}{4}\right)^{k} u[k] + 5(3)^{k} u[-(k+1)]$$

Answer: $\left[-(\frac{1}{4})^k + 3(\frac{1}{2})^k\right] u[k] + 6(3)^k u[-(k+1)] \quad \bigtriangledown$

11.8 Summary

In this chapter we discuss the analysis of linear, time-invariant, discrete-time (LTID) systems by z-transform. The z-transform is an extension of the DTFT with the frequency variable $j\Omega$ generalized to $\sigma + j\Omega$. Such an extension allows us to synthesize discrete-time signals by using exponentially growing (discrete-time) sinusoids. The relationship of the z-transform to the DTFT is identical to that of the Laplace transform to the Fourier. Because of the generalization of the frequency variable, we can analyze all kinds of LTID systems and also handle exponentially growing inputs.

The z-transform changes the difference equations of LTID systems into algebraic equations. Therefore, solving these difference equations reduces to solving algebraic equations.

The transfer function H[z] of an LTID system is equal to the ratio of the ztransform of the output to the z-transform of the input when all initial conditions are zero. Therefore, if F[z] is the z-transform of the input f[k] and Y[z] is the z-transform of the corresponding output y[k] (when all initial conditions are zero), then Y[z] = H[z]F[z]. For a system specified by the difference equation Q[E]y[k] =P[E]f[k], the transfer function H[z] = P[z]/Q[z]. Moreover, H[z] is the z-transform of the system impulse response h[k]. We also showed in Chapter 9 that the system response to an everlasting exponential z^k is $H[z]z^k$.

LTID systems can be realized by scalar multipliers, summers, and time delays. A given transfer function can be synthesized in many different ways. Canonical, cascade and parallel forms of realization are discussed. The realization procedure is identical to that for continuous-time systems.

In Sec. 11.5, we showed that discrete-time systems can be analyzed by the Laplace transform as if they were continuous-time systems. In fact, we showed that the z-transform is the Laplace transform with a change in variable.

In practice, we often have to deal with hybrid systems consisting of discretetime and continuous-time subsystems. Feedback hybrid systems are also called sampled-data systems. In such systems, we can relate the samples of the output to those of the input. However, the output is generally a continuous-time signal. The output values during the successive sampling intervals can be found by using the modified z-transform.

The majority of the input signals and practical systems are causal. Consequently, we are required to deal with causal signals most of the time. When all

Problems

signals are restricted to the causal type, the z-transform analysis is greatly simplified; the region of convergence of a signal becomes irrelevant to the analysis process. This special case of z-transform (which is restricted to causal signals) is called the unilateral z-transform. Much of the chapter deals with this transform. Section 11.7 discusses the general variety of the z-transform (bilateral z-transform), which can handle causal and noncausal signals and systems. In the bilateral transform, the inverse transform of F[z] is not unique, but depends on the region of convergence of F[z]. Thus, the region of convergence plays a crucial role in the bilateral z-transform.

Problems

11.1-1 Using the definition of the z-transform, show that

(a)
$$\gamma^{k-1}u[k-1] \iff \frac{1}{z-\gamma}$$
 (c) $\frac{\gamma^k}{k!}u[k] \iff e^{\gamma/z}$
(b) $u[k-m] \iff \frac{z}{z^m(z-1)}$ (d) $\frac{(\ln \alpha)^k}{k!}u[k] \iff \alpha^{1/z}$

11.1-2 Using only the z-transform Table 11.1, show that

(a) $2^{k+1}u[k-1] + e^{k-1}u[k] \iff \frac{4}{z-2} + \frac{z}{e(z-e)}$ (b) $k\gamma^k u[k-1] \iff \frac{\gamma z}{(z-\gamma)^2}$ Hint: Express u[k-1] in terms of u[k]. (c) $\left[2^{-k}\cos\left(\frac{\pi}{3}k\right)\right]u[k-1] \iff \frac{0.25(z-1)}{z^2-0.5z+0.25}$ Hint: See the hint for part b. (d) $k(k-1)(k-2)2^{k-3}u[k-m] \iff \frac{6z}{z}$ for

(d) $k(k-1)(k-2)2^{k-3}u[k-m] \iff \frac{6z}{(z-2)^4}$ for m=0, 1, 2, or 3. Hint: Examine what happens to the function if u[k-m] is replaced by u[k].

11.1-3 Find the inverse z-transform of

(a)
$$\frac{z(z-4)}{z^2-5z+6}$$
 (g) $\frac{z(z-2)}{z^2-z+1}$
(b) $\frac{z-4}{z^2-5z+6}$ (h) $\frac{2z^2-0.3z+0.2z}{z^2+0.6z+0.25}$

(c)
$$\frac{(e^{-2}-2)z}{(z-e^{-2})(z-2)}$$
 (i) $\frac{2z(3z-23)}{(z-1)(z^2-6z+25)}$

(d)
$$\frac{z(2z+3)}{(z-1)(z^2-5z+6)}$$
 (j) $\frac{z(3.83z+11.34)}{(z-2)(z^2-5z+25)}$

$$\frac{-5z+22)}{(1)(z-2)^2}$$
(k) $\frac{z^2(-2z^2+8z-7)}{(z-1)(z-2)^3}$
(1.4z+0.08)

(f)
$$\frac{1}{(z-0.2)(z-0.8)^2}$$

11.1-4 Find the first three terms of
$$f[k]$$
 if

$$F[z] = \frac{2z^3 + 13z^2 + z}{z^3 + 7z^2 + 2z + 1}$$





Fig. P11.2-1

Fig. P11.2-2

Find your answer by expanding F[z] as a power series in z^{-1} .

11.1-5 By expanding

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$$F[z] = rac{\gamma z}{(z-\gamma)^2}$$

as a power series in z^{-1} , show that $f[k] = k\gamma^k u[k]$.

11.2-1 For a discrete-time signal shown in Fig. P11.2-1 show that

$$F[z] = \frac{1 - z^{-m}}{1 - z^{-1}}$$

- 11.2-2 Find the z-transform of the signal illustrated in Fig. P11.2-2. Solve this problem in two ways, as in Examples 11.2d and 11.4. Verify that the two answers are equivalent.
- 11.2-3 Using only the fact that $\gamma^k u[k] \iff \frac{z}{z-\gamma}$ and properties of the z-transform, find the z-transform of

(a)
$$k^2 \gamma^k u[k]$$
 (c) $a^k [u[k] - u[k - m]]$
(b) $k^3 u[k]$ (d) $k e^{-2k} u[k - m]$

- 11.2-4 Using only Pair 1 in Table 11.1 and appropriate properties of the z-transform, derive iteratively pairs 2 through 9. In other words, first derive Pair 2. Then use Pair 2 (and Pair 1, if needed) to derive Pair 3, and so on. However, pair 6 should be derived after pair 7.
- 11.3-1 Solve Prob. 9.4-9 by the z-transform method.

11.3-2 Solve

$$y[k+1] + 2y[k] = f[k+1]$$

with y[0] = 1 and $f[k] = e^{-(k-1)}u[k]$

11.3-3 Find the output y[k] of an LTID system specified by the equation

$$2y[k+2] - 3y[k+1] + y[k] = 4f[k+2] - 3f[k+1]$$

if the initial conditions are y[-1] = 0, y[-2] = 1, and the input $f[k] = (4)^{-k}u[k]$.

- 11.3-4 Solve Prob. 11.3-3 if instead of initial conditions y[-1], y[-2] you are given the auxiliary conditions $y[0] = \frac{3}{2}$ and $y[1] = \frac{35}{4}$.
- 11.3-5 Solve

$$4y[k+2] + 4y[k+1] + y[k] = f[k+1]$$

with y[-1] = 0, y[-2] = 1, and f[k] = u[k]. 11.3-6 Solve y[k+2] - 3y[k+1] + 2y[k] = f[k+1] Problems

if y[-1] = 2, y[-2] = 3, and $f[k] = (3)^k u[k]$.

11.3-7 Solve

$$y[k+2] - 2y[k+1] + 2y[k] = f[k]$$

with y[-1] = 1, y[-2] = 0, and f[k] = u[k].

11.3-8 Solve

$$y[k] + 2y[k-1] + 2y[k-2] = f[k-1] + 2f[k-2]$$

with y[0] = 0, y[1] = 1, and $f[k] = e^k u[k]$.

11.3-9 (a) Find the zero-state response of an LTID system with transfer function

$$H[z] = \frac{z}{(z+0.2)(z-0.8)}$$

and the input $f[k] = e^{(k+1)}u[k]$.

(b) Write the difference equation relating the output y[k] to input f[k].

11.3-10 Repeat Prob. 11.3-9 if f[k] = u[k] and

$$H[z] = \frac{2z+3}{(z-2)(z-3)}$$

11.3-11 Repeat Prob. 11.3-9 if

$$H[z] = \frac{6(5z-1)}{6z^2 - 5z + 1}$$

and the input f[k] is (a) $(4)^{-k}u[k]$ (b) $(4)^{-(k-2)}u[k-2]$ (c) $(4)^{-(k-2)}u[k]$ (d) $(4)^{-k}u[k-2]$.

11.3-12 Repeat Prob. 11.3-9 if f[k] = u[k] and

$$H[z] = \frac{2z - 1}{z^2 - 1.6z + 0.8}$$

- 11.3-13 Find the transfer functions corresponding to each of the systems specified by difference equations in Probs. 11.3-2, 11.3-3, 11.3-5, and 11.3-8.
- 11.3-14 Find h[k], the unit impulse response of the systems described by the following equations:
 - (a) y[k] + 3y[k-1] + 2y[k-2] = f[k] + 3f[k-1] + 3f[k-2]

(b) y[k+2] + 2y[k+1] + y[k] = 2f[k+2] - f[k+1]

(c) y[k] - y[k-1] + 0.5y[k-2] = f[k] + 2f[k-1]

11.3-15 Find h[k], the unit impulse response of the systems in Probs. 11.3-9, 11.3-10, and 11.3-12.

11.4-1 Show a canonical, a cascade and a parallel realization of the following transfer functions:

(a)
$$H[z] = \frac{z(3z-1.3)}{z^2 - z + 0.16}$$

(b) $H[z] = \frac{5z + 2.2}{z^2 + z + 0.16}$
(c) $H[z] = \frac{3.8z - 1.1}{(z - 0.2)(z^2 - 0.6z + 0.25)}$

11.4-2 Give cascade and parallel realizations of the following transfer functions:

(a)
$$\frac{z(1.6z-1.8)}{(z-0.2)(z^2+z+0.5)}$$
 (b) $\frac{z(2z^2+1.3z+0.96)}{(z+0.5)(z-0.4)^2}$