DYNAMICS OF LINEAR SYSTEMS

3.1 DIFFERENTIAL EQUATIONS REVISITED

In the last chapter we saw that the dynamic behavior of many dynamic systems is quite naturally characterized by systems of first-order differential equations. For a general system these equations in state space notation take the form

$$\dot{x} = f(x, u, t)$$

and in a linear system they take the special form

$$\dot{x} = A(t)x + B(t)u \tag{3.1}$$

where $x = [x_1, x_2, \ldots, x_k]'$ is the system state vector and $u = [u_1, u_2, \ldots, u_m]'$ is the input vector.

If the matrices A and B in (3.1) are constant matrices, i.e., not functions of time, the system is said to be "time-invariant." Time-varying systems are conceptually and computationally more difficult to handle than time-invariant systems. For this reason our attention will be devoted primarily to time-invariant systems. Fortunately many processes of interest can be approximated by linear, time-invariant models.

In using the conventional, frequency-domain approach the differential equations are converted to transfer functions as soon as possible, and the dynamics of a system comprising several subsystems is obtained by combining the transfer functions of the subsystems using well-known techniques (reviewed in Chap. 4). With the state-space methods, on the other hand, the description of the system dynamics in the form of differential equations is retained throughout the analysis and design. In fact, if a subsystem is characterized by a transfer

58

function it is often necessary to convert the transfer fun[...] equations in order to proceed by state-space methods.

In this chapter we shall develop the general formula for the solution of a vector-matrix differential equation in the form of (3.1) in terms of a very important matrix known as the *state-transition matrix* which describes how the state $x(t)$ of the system at some time t evolves into (or from) the state $x(\tau)$ at some other time τ. For time-invariant systems, the state-transition matrix is the matrix exponential function, which is easily calculated. For most time-varying systems, however, the state-transition matrix, although known to exist, cannot be expressed in terms of simple functions (such as real or complex exponentials) or even not-so-simple functions (such as Bessel functions, hypergeometric functions). Thus, while many of the results developed for time-invariant systems apply to time-varying systems, it is very difficult as a practical matter to carry out the required calculations. This is one reason why our attention is confined mainly (but not exclusively) to time-invariant systems. The world of real applications contains enough of the latter to keep a design engineer occupied.

3.2 SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS IN STATE-SPACE FORM

Time-invariant dynamics The simplest form of the general differential equation of the form (3.1) is the "homogeneous," i.e., unforced equation

$$\dot{x} = Ax \tag{3.2}$$

where A is a constant k by k matrix. The solution to (3.2) can be expressed as

$$x(t) = e^{At}c \tag{3.3}$$

where e^{At} is the matrix exponential function

$$e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + \cdots \tag{3.4}$$

and c is a suitably chosen constant vector. To verify (3.3) calculate the derivative of $x(t)$

$$\frac{dx(t)}{dt} = \frac{d}{dt}(e^{At})c \tag{3.5}$$

and, from the defining series (3.4),

$$\frac{d}{dt}(e^{At}) = A + A^2 t + A^3\frac{t^2}{2!} + \cdots = A\left(I + At + A^2\frac{t^2}{2!} + \cdots\right) = A e^{At}$$

Thus (3.5) becomes

$$\frac{dx(t)}{dt} = Ae^{At}c = Ax(t)$$

which was to be shown. To evaluate the constant c suppose that at some time τ the state $x(\tau)$ is given. Then, from (3.3),

$$x(\tau) = e^{A\tau}c \tag{3.6}$$

Multiplying both sides of (3.6) by the inverse of $e^{A\tau}$ we find that

$$c = (e^{A\tau})^{-1}x(\tau)$$

Thus the general solution to (3.2) for the state $x(t)$ at time t, given the state $x(\tau)$ at time τ, is

$$x(t) = e^{At}(e^{A\tau})^{-1}x(\tau) \tag{3.7}$$

The following property of the matrix exponential can readily be established by a variety of methods—the easiest perhaps being the use of the series definition (3.4)—

$$e^{A(t_1+t_2)} = e^{At_1}e^{At_2} \tag{3.8}$$

for any t_1 and t_2. From this property it follows that

$$(e^{A\tau})^{-1} = e^{-A\tau} \tag{3.9}$$

and hence that (3.7) can be written

$$x(t) = e^{A(t-\tau)}x(\tau) \tag{3.10}$$

The matrix $e^{A(t-\tau)}$ is a special form of the *state-transition matrix* to be discussed subsequently.

We now turn to the problem of finding a "particular" solution to the nonhomogeneous, or "forced," differential equation (3.1) with A and B being constant matrices. Using the "method of the variation of the constant,"[1] we seek a solution to (3.1) of the form

$$x(t) = e^{At}c(t) \tag{3.11}$$

where $c(t)$ is a function of time to be determined. Take the time derivative of $x(t)$ given by (3.11) and substitute it into (3.1) to obtain:

$$Ae^{At}c(t) + e^{At}\dot{c}(t) = Ae^{At}c(t) + Bu(t)$$

or, upon cancelling the terms $A e^{At}c(t)$ and premultiplying the remainder by e^{-At},

$$\dot{c}(t) = e^{-At}Bu(t) \tag{3.12}$$

Thus the desired function $c(t)$ can be obtained by simple integration (the mathematician would say "by a quadrature")

$$c(t) = \int_T^t e^{-A\lambda}Bu(\lambda)\,d\lambda$$

The lower limit T on this integral cannot as yet be specified, because we will need to put the particular solution together with the solution to the

homogeneous equation to obtain the complete (general) solution. For the present, let T be undefined. Then the particular solution, by (3.11), is

$$x(t) = e^{At} \int_T^t e^{-A\lambda} Bu(\lambda) \, d\lambda = \int_T^t e^{A(t-\lambda)} Bu(\lambda) \, d\lambda \qquad (3.13)$$

In obtaining the second integral in (3.13), the exponential e^{At}, which does not depend on the variable of integration λ, was moved under the integral, and property (3.8) was invoked to write $e^{At}e^{-A\lambda} = e^{A(t-\lambda)}$.

The complete solution to (3.1) is obtained by adding the "complementary solution" (3.10) to the particular solution (3.13). The result is

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_T^t e^{A(t-\lambda)} Bu(\lambda) \, d\lambda \qquad (3.14)$$

We can now determine the proper value for lower limit T on the integral. At $t = \tau$ (3.14) becomes

$$x(\tau) = x(\tau) + \int_T^\tau e^{A(t-\lambda)} Bu(\lambda) \, d\lambda \qquad (3.15)$$

Thus, the integral in (3.15) must be zero for any $u(t)$, and this is possible only if $T = \tau$. Thus, finally we have the complete solution to (3.1) when A and B are constant matrices

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-\lambda)} Bu(\lambda) \, d\lambda \qquad (3.16)$$

This important relation will be used many times in the remainder of the book. It is worthwhile dwelling upon it. We note, first of all, that the solution is the sum of two terms: the first is due to the "initial" state $x(\tau)$ and the second—the integral—is due to the input $u(\tau)$ in the time interval $\tau \leq \lambda \leq t$ between the "initial" time τ and the "present" time t. The terms initial and present are enclosed in quotes to denote the fact that these are simply convenient definitions. There is no requirement that $t \geq \tau$. The relationship is perfectly valid even when $t \leq \tau$.

Another fact worth noting is that the integral term, due to the input, is a "convolution integral": the contribution to the state $x(t)$ due to the input u is the convolution of u with $e^{At}B$. Thus the function $e^{At}B$ has the role of the impulse response[1] of the system whose output is $x(t)$ and whose input is $u(t)$.

If the output y of the system is not the state x itself but is defined by the observation equation

$$y = Cx$$

then this output is expressed by

$$y(t) = C e^{A(t-\tau)}x(t) + \int_\tau^t C e^{A(t-\lambda)} Bu(\lambda) \, d\lambda \qquad (3.17)$$

and the impulse response of the system with y regarded as the output is $C e^{A(t-\lambda)}B$.

The development leading to (3.16) and (3.17) did not really require that B and C be constant matrices. By retracing the steps in the development it is readily seen that when B and C are time-varying, (3.16) and (3.17) generalize to

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} e^{A(t-\lambda)}B(\lambda)u(\lambda) \, d\lambda \qquad (3.18)$$

and

$$y(t) = C(t) \, e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} C(t) \, e^{A(t-\lambda)}B(\lambda)u(\lambda) \, d\lambda \qquad (3.19)$$

Time-varying dynamics Unfortunately, however, the results expressed by (3.18) and (3.19) do *not* hold when A is time-varying.

In any unforced (homogeneous) system the state at time t depends only on the state at time τ. In a linear system, this dependence is linear; thus we can always write the solution to $\dot{x} = A(t)x$ as

$$x(t) = \Phi(t, \tau)x(\tau) \qquad (3.20)$$

The matrix $\Phi(t, \tau)$ that relates the state at time t to the state at time τ is generally known as the *state-transition matrix* because it defines how the state $x(\tau)$ evolves (or "transitions") into (or from) the state $x(t)$. In a time-invariant system $\Phi(t, \tau) = e^{A(t-\tau)}$, but there is no simple expression for the state-transition matrix in a time-varying system. The absence of such an expression is rarely a serious problem, however. It is usually possible to obtain a control system design from only a knowledge of the dynamics matrix $A(t)$, without having an expression for the transition matrix.

The complete solution to (3.1) can be expressed in the form of (3.18), with the general transition matrix $\Phi(t, \tau)$ replacing the matrix exponential of a time-invariant system. The general solution is thus given by

$$x(t) = \Phi(t, \tau)x(\tau) + \int_{\tau}^{t} \Phi(t, \lambda)B(\lambda)u(\lambda) \, d\lambda \qquad (3.21)$$

$$y(t) = C(t)\Phi(t, \tau)x(\tau) + \int_{\tau}^{t} C(t)\Phi(t, \lambda)B(\lambda)u(\lambda) \, d\lambda \qquad (3.22)$$

The derivation of (3.21) follows the same pattern as was used to obtain (3.18). The reader might wish to check his comprehension of the development by deriving (3.21). The development can also be found in a number of textbooks on linear systems, [1] for example.

The state-transition matrix The state-transition matrix for a time-invariant system can be calculated by various methods. One of these is to use the series definition (3.4) as will be illustrated in Example 3A. This is generally not a

convenient method for pencil-and-paper calculations. It sometimes may be appropriate for numerical calculations, although there are better methods. (See Note 3.1.) For pencil-and-paper calculations, the Laplace transform method, to be developed in Sec. 3.4, is about as good a method as any.

It should be noted that the state-transition matrix for a time-invariant system is a function only of the *difference* $t - \tau$ between the initial time τ and the present time t as would be expected for a time-invariant system. (See Note 3.2.) Thus, in a time-invariant system, there is no loss in generality in taking the initial time τ to be zero and in computing $\Phi(t) = e^{At}$. If, for a subsequent calculation the initial time is not zero, and $\Phi(t, \tau)$ is needed, it is obtained from $\Phi(t)$ by replacing t by $t - \tau$.

In a time-varying system this procedure is of course not valid; both the initial time and the present time must be treated as general variables. A knowledge of $\Phi(t, 0)$ is not adequate information for the determination of $\Phi(t, \tau)$.

Although the state transition matrix cannot be calculated analytically in general, it is *sometimes* possible to do so because of the very simple structure of the dynamics matrix $A(t)$, as will be illustrated in the missile-guidance example below. Thus, if an application arises in which an expression is necessary for the transition matrix of a time-varying system, the engineer should consider "having a go at it," using whatever ad hoc measures appear appropriate.

Example 3A Motion of mass without friction The differential equation for the position of a mass to which an external force f is applied is

$$\ddot{x} = f/m = u \qquad (3A.1)$$

(The control variable $u = f/m$ in this case is the total acceleration.)

Defining the state variables by

$$x_1 = x \qquad x_2 = \dot{x}$$

results in the state-space form

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u \qquad (3A.2)$$

Thus, for this example,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using the series definition (3.4) we obtain the state transition matrix

$$\Phi(t) = e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

The series terminates after only two terms.

The integral in (3.18) with $\tau = 0$ is given by

$$\int_0^t \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\lambda)\, d\lambda = \begin{bmatrix} \int_0^t \lambda u(\lambda)\, d\lambda \\ \int_0^t u(\lambda)\, d\lambda \end{bmatrix}$$

Thus, the solution to (3A.2), using the general formula (3.18) is given by

$$x_1(t) = x_1(0) + tx_2(0) + \int_0^t \lambda u(\lambda)\, d\lambda$$

$$x_2(t) = x_2(0) + \int_0^t u(\lambda)\, d\lambda$$

Obviously these answers could have been obtained directly from (3A.1) without using all the state-space apparatus being developed. This apparatus has its greatest utility when simple methods fail.

Example 3B Missile guidance The equations of motion (assumed to be confined to a plane) of a missile moving at constant speed, relative to a target also moving at constant speed, can be approximated by

$$\dot{\lambda} = \frac{1}{V\bar{T}^2} z$$

$$\dot{z} = \bar{T} u \tag{3B.1}$$

where λ is the line-of-sight angle to the target

$\quad z$ is the projected miss distance

$\quad V$ is the velocity of the missile relative to the target

$\quad \bar{T} = T - t$ is the "time-to-go"

$\quad u$ is the acceleration normal to the missile relative velocity vector

It is assumed that the terminal time T is a known quantity. (The reader should review the discussion in Prob. 2.6 for the significance of these variables and the derivation of (3B.1).)

Using the state-variable definitions

$$x_1 = \lambda \qquad x_2 = z$$

results in the matrices

$$A(t) = \begin{bmatrix} 0 & \dfrac{1}{V\bar{T}^2} \\ 0 & 0 \end{bmatrix} \qquad B(t) = \begin{bmatrix} 0 \\ \bar{T} \end{bmatrix} \tag{3B.2}$$

Since $A(t)$ is time-varying (through $\bar{T}$), the transition matrix is *not* the matrix exponential and cannot be found using the series (3.4). In this case, however, we can find the transition matrix by an ad hoc method. First we note that the transition matrix $\Phi(t, \tau)$ expresses the solution to the unforced system

$$\dot{\lambda} = \frac{1}{V\bar{T}^2} z \tag{3B.3}$$

$$\dot{z} = 0 \tag{3B.4}$$

The general form of this solution is

$$\lambda(t) = \phi_{11}(t, \tau)\lambda(\tau) + \phi_{12}(t, \tau)z(\tau)$$

$$z(t) = \phi_{21}(t, \tau)\lambda(\tau) + \phi_{22}(t, \tau)z(\tau) \tag{3B.5}$$

The terms $\phi_{ij}(t, \tau)$ $(i, j) = 1, 2$, which we will now calculate, are the elements of the required transition matrix.

From (3B.4) we have immediately

$$z(t) = z(\tau) = \text{const} \tag{3B.6}$$

Hence

$$\phi_{21}(t, \tau) = 0 \qquad \phi_{22}(t, \tau) = 1 \tag{3B.7}$$

The easiest way to get the first row (ϕ_{11} and ϕ_{12}) of the transition matrix is to use (3B.3) which can be written

$$V(T - \xi)^2 \dot{\lambda}(\xi) = z(\xi) \qquad \text{for all } \xi$$

Thus

$$V(T - \tau)^2 \lambda(\tau) = z(\tau)$$

But, from (3B.6), $z(\xi) = z(\tau)$. Hence

$$\dot{\lambda}(\xi) = \frac{1}{(T - \xi)^2} z(\tau) \tag{3B.8}$$

Integrate both sides of (3B.8) from τ to t

$$\int_\tau^t \dot{\lambda}(\xi) \, d\xi = \int_\tau^t \frac{d\xi}{(T - \xi)^2} z(\tau) \, d\xi$$

or

$$\lambda(t) - \lambda(\tau) = \left(\frac{1}{T - t} - \frac{1}{T - \tau} \right) z(\tau) \tag{3B.9}$$

Thus, from (3B.9), we obtain

$$\phi_{11}(t, \tau) = 1 \qquad \phi_{12}(t, \tau) = \frac{1}{T - t} - \frac{1}{T - \tau} \tag{3B.10}$$

Combining (3B.10) with (3B.7) gives the state transition matrix

$$\Phi(t, \tau) = \begin{bmatrix} 1 & \dfrac{1}{T - t} - \dfrac{1}{T - \tau} \\ 0 & 1 \end{bmatrix} \tag{3B.11}$$

3.3 INTERPRETATION AND PROPERTIES OF THE STATE-TRANSITION MATRIX

The state-transition matrix, which is fundamental to the theory of linear dynamic systems, has a number of important properties which are the subject of this section.

We note, first of all, that the state-transition matrix is an expression of the solution to the homogeneous equation

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{3.23}$$

where $x(t)$ is given by (3.20). The time derivative of $x(t)$ in (3.20) must of course satisfy (3.23) for any t and $x(t)$. In (3.20) $x(\tau)$ represents initial data and is not a time function. Thus

$$\frac{dx(t)}{dt} = \frac{\partial \Phi(t, \tau)}{\partial t} x(\tau) \tag{3.24}$$

(Since the transition matrix is a function of two arguments t and τ, it is necessary to write its time derivative as a *partial* derivative. The transition matrix also has a derivative with respect to the "initial" time τ which is investigated in Prob. 3.4.) Substitution of (3.24) and (3.20) into (3.23) gives

$$\frac{\partial \Phi(t, \tau)}{\partial t} x(\tau) = A(t)\Phi(t, \tau)x(\tau)$$

Since this must hold for any $x(\tau)$, we may cancel $x(\tau)$ on both sides to finally obtain

$$\frac{\partial \Phi(t, \tau)}{\partial t} = A(t)\Phi(t, \tau) \tag{3.25}$$

In other words, the transition matrix Φ satisfies the same differential equation as the state x. This can be emphasized by writing (3.25) simply as

$$\dot{\Phi} = A\Phi \tag{3.26}$$

which does not explicitly exhibit the time dependence of A and Φ. The dot on top of Φ must be interpreted to designate differentiation with respect to the first argument. (Because of the possibility of confusion of arguments use of the full expression (3.25) is recommended in analytical studies.)

We note that (3.20) holds for any t and τ, including $t = \tau$. Thus

$$x(t) = \Phi(t, t)x(t)$$

for any $x(t)$. Thus we conclude that

$$\Phi(t, t) = I \qquad \text{for any } t \tag{3.27}$$

This becomes the initial condition for (3.25) or (3.26).

Other properties of the transition matrix follow from the fact that the differential equation (3.23) not only possesses a solution for any initial state $x(\tau)$ and any time interval $[\tau, t]$ but that this solution is unique. This is a basic theorem in the theory of ordinary differential equations and is proved in standard textbooks on the subject, e.g., [2, 3]. There are certain restrictions on the nature of permissible time variations of $A(t)$ but these are always satisfied in real-world systems. When A is a constant matrix, of course, not only do we know that Φ exists but we have an expression for it, namely $\Phi(t) = e^{At}$.

Assuming the existence and uniqueness of solutions, we can write

$$x(t_3) = \Phi(t_3, t_1)x(t_1) \qquad \text{for any } t_3, t_1 \tag{3.28}$$

and also

$$x(t_3) = \Phi(t_3, t_2)x(t_2) \qquad \text{for any } t_3, t_2 \tag{3.29}$$

$$x(t_2) = \Phi(t_2, t_1)x(t_1) \qquad \text{for any } t_2, t_1 \tag{3.30}$$

Thus, substituting (3.30) into (3.29)

$$x(t_3) = \Phi(t_3, t_2)\Phi(t_2, t_1)x(t_1) \tag{3.31}$$

Comparing (3.31) with (3.28) we see that

$$\Phi(t_3, t_1) = \Phi(t_3, t_2)\Phi(t_2, t_1) \qquad \text{for any } t_3, t_2, t_1 \qquad (3.32)$$

This very important property—known as the *semigroup* property—of the state-transition matrix is a direct consequence of the fact that whether we go from state $x(t_1)$ to $x(t_3)$ directly or via an "intermediate" state $x(t_2)$, we must end at the same point. Note, however, that the time t_2 of the intermediate state need not be between t_1 and t_3.

The semigroup properties (3.32) and (3.27) gives

$$I = \Phi(t, \tau)\Phi(\tau, t)$$

or

$$\Phi(\tau, t) = [\Phi(t, \tau)]^{-1} \qquad \text{for any } t, \tau \qquad (3.33)$$

This of course means that the state-transition matrix is never singular even if the dynamics matrix A is singular, as it often is.

In a time-invariant system, the transition matrix is characterized by a single argument, as already discussed:

$$\Phi(t_1, t_2) = \Phi(t_1 - t_2)$$

Thus, for time-invariant systems, the properties (3.27), (3.32), and (3.33) become

$$\Phi(0) = I \qquad (3.34)$$

$$\Phi(t)\Phi(\tau) = \Phi(t + \tau) \qquad (3.35)$$

$$\Phi^{-1}(t) = \Phi(-t) \qquad (3.36)$$

It is readily verified that $\Phi(t) = e^{At}$ possesses these properties:

$$e^{A0} = I \qquad (3.37)$$

$$e^{At} e^{A\tau} = e^{A(t+\tau)} \qquad (3.38)$$

$$(e^{At})^{-1} = e^{-At} \qquad (3.39)$$

The first relation (3.37) is apparent from the series definition (3.4) and the second relation (3.38) can be verified by multiplying the series for e^{At} by the series for $e^{A\tau}$. (The calculations are a bit tedious, but the skeptical reader is invited to perform them.) The third relation (3.39) follows from the first two.

By analogy with (3.38) the reader might be tempted to conclude that $e^{At}e^{Bt} = e^{(A+B)t}$. This is generally *not* true, however. In order for it to be true A and B must commute (i.e., $AB = BA$) and this condition is rarely met in practice.

3.4 SOLUTION BY THE LAPLACE TRANSFORM: THE RESOLVENT

As the reader is no doubt aware, Laplace transforms are very useful for solving time-invariant differential equations. Indeed Laplace transforms are the basis of the entire frequency-domain methodology, to which the next chapter is devoted.

The Laplace transform of a signal $f(t)$ which may be an input variable or a state variable is defined by

$$\mathcal{L}[f(t)] = \mathsf{f}(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{3.40}$$

where s is a complex variable generally called *complex frequency*. A discussion of the region of convergence of $\mathsf{f}(s)$ in the complex s plane, and many other details about the Laplace transform are to be found in many standard textbooks such as [1] and [4].

The sans-serif letter f used to designate the Laplace transform of $f(t)$ was chosen advisedly. In texts in which the signals are all scalars, capital letters are used to denote Laplace transforms (viz., $X(s) = \mathcal{L}[x(t)]$, $Y(s) = \mathcal{L}[y(t)]$, etc.). But in this book capital letters have been preempted for designating matrices. The use of sans-serif letters for Laplace transforms avoids the risk of confusion.

The lower limit on the integral has been written as 0. In accordance with engineering usage, this is understood to be 0^-, that is, the instant just prior to the occurrence of discontinuities, impulses, etc., in the signals under examination. The reader who is unfamiliar with this usage should consult a standard text such as [1] or [4].

The Laplace transform is useful for solving (3.1) only when A and B are constant matrices, which we will henceforth assume. In order to use the Laplace transform, we need an expression for the Laplace transform of the time derivative of $f(t)$

$$\mathcal{L}[\dot{f}(t)] = \int_0^\infty e^{-st}\frac{df}{dt}\,dt = e^{-st}f(t)\Big|_0^\infty - \int_0^\infty -s\,e^{-st}f(t)\,dt \tag{3.41}$$

upon integration by parts. Assuming

$$\lim_{t\to\infty} e^{-st}f(t) \to 0$$

(3.41) becomes

$$\mathcal{L}[\dot{f}(t)] = s\int_0^\infty e^{-st}f(t)\,dt - f(0) = s\mathsf{f}(s) - f(0) \tag{3.42}$$

We also note that (3.42) applies when $f(t)$ is a vector:

$$\mathcal{L}[f(t)] = \mathcal{L}\begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix} = \begin{bmatrix} \mathcal{L}[f_1(t)] \\ \vdots \\ \mathcal{L}[f_n(t)] \end{bmatrix} = \begin{bmatrix} \mathsf{f}_1(s) \\ \vdots \\ \mathsf{f}_n(s) \end{bmatrix} = \mathsf{f}(s) \tag{3.43}$$

and also that

$$\mathscr{L}[Ax(t)] = Ax(s) \qquad (3.44)$$

Applying all of these to (3.1) with A and B constant gives

$$sx(s) - x(0) = Ax(s) + Bu(s)$$

or

$$(sI - A)x(s) = x(0) + Bu(s)$$

Solve for $x(s)$ to obtain

$$x(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}Bu(s) \qquad (3.45)$$

On taking the inverse Laplace transform of $x(s)$ as given by (3.45) we obtain the desired solution for $x(t)$. We note that $x(s)$ is the sum of two terms, the first due to the initial condition $x(0)$ multiplied by the matrix $(sI - A)^{-1}$ and the second being the product of this matrix and the term due to the input $Bu(s)$. Knowing the inverse Laplace transform of $(sI - A)^{-1}$ would permit us to find the inverse Laplace transform of (3.45) and hence obtain $x(t)$. In the scalar case we recall that

$$\mathscr{L}[e^{at}] = \frac{1}{s - a} = (s - a)^{-1} \qquad (3.46)$$

We have not yet discussed calculating the Laplace transform of a matrix function of time. But we should not be very much surprised to learn that

$$\mathscr{L}[e^{At}] = (sI - A)^{-1} \qquad (3.47)$$

which is simply the matrix version of (3.46). It can be shown by direct calculation (see Note 3.3) that (3.47) is in fact true. And if this be the case then the inverse Laplace transform of (3.45) is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \qquad (3.48)$$

which is the desired solution. The integral term in (3.48) is given by the well-known *convolution theorem* for the Laplace transform [1]

$$\mathscr{L}\left[\int_0^t f(t - \lambda)g(\lambda)\,d\lambda\right] = f(s)g(s)$$

which is readily extended from scalar functions to matrices.

The solution for $x(t)$ given by (3.48) is a special case (namely $\tau = 0$) of the general solution (3.16) obtained by another method of analysis. This confirms, if confirmation is necessary, the validity of (3.47).

The exponential matrix e^{At} is known as the state transition matrix (for a time invariant system) and its Laplace transform

$$\Phi(s) = (sI - A)^{-1} \qquad (3.49)$$

is known in mathematical literature as the *resolvent* of A. In engineering literature this matrix has been called the *characteristic frequency* matrix[1] or simply the *characteristic matrix*.[4] Regrettably there doesn't appear to be a standard symbol for the resolvent, which we have designated as $\Phi(s)$ in this book.

The fact that the state transition matrix is the inverse Laplace transform of the resolvent matrix facilitates the calculation of the former. It also characterizes the dynamic behavior of the system, the subject of the next chapter. The steps one takes in calculating the state-transition matrix using the resolvent are:

(*a*) Calculate $sI - A$.
(*b*) Obtain the resolvent by inverting $(sI - A)$.
(*c*) Obtain the state-transition matrix by taking the inverse Laplace transform of the resolvent, element by element.

The following examples illustrate the process.

Example 3C DC motor with inertial load In Chap. 2 (Example 2B) we found that the dynamics of a dc motor driving an inertial load are

$$\dot{\theta} = \omega$$

$$\dot{\omega} = -\alpha\omega + \beta u$$

The matrices of the state-space characterization are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

Thus the resolvent is

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+\alpha \end{bmatrix}^{-1} = \frac{1}{s(s+\alpha)}\begin{bmatrix} s+\alpha & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s(s+\alpha)} \\ 0 & \dfrac{1}{s+\alpha} \end{bmatrix}$$

Finally, taking the inverse Laplace transforms of each term in $\Phi(s)$ we obtain

$$e^{At} = \Phi(t) = \begin{bmatrix} 1 & (1 - e^{-\alpha t})/\alpha \\ 0 & e^{-\alpha t} \end{bmatrix}$$

Example 3D Inverted pendulum The equations of motion of an inverted pendulum were determined to be (approximately)

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \Omega^2\theta + u$$

Hence the matrices of the state-space characterization are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The resolvent is

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ -\Omega^2 & s \end{bmatrix}^{-1} = \frac{1}{s^2 - \Omega^2}\begin{bmatrix} s & 1 \\ \Omega^2 & s \end{bmatrix}$$

and the state-transition matrix is

$$\Phi(t) = e^{At} = \begin{bmatrix} \cosh \Omega t & \sinh \Omega t / \Omega \\ \Omega \sinh \Omega t & \cosh \Omega t \end{bmatrix}$$

For a general kth-order system the matrix $sI - A$ has the following appearance

$$sI - A = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1k} \\ -a_{21} & s - a_{22} & \cdots & -a_{2k} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -a_{k1} & -a_{k2} & \cdots & s - a_{kk} \end{bmatrix} \qquad (3.50)$$

We recall (see Appendix) that the inverse of any matrix M can be written as the adjoint matrix, adj M, divided by the determinant $|M|$. Thus

$$(sI - A)^{-1} = \frac{\text{adj }(sI - A)}{|sI - A|}$$

If we imagine calculating the determinant $|sI - A|$ we see that one of the terms will be the product of the diagonal elements of $sI - A$:

$$(s - a_{11})(s - a_{22}) \cdots (s - a_{kk}) = s^k + c_1 s^{k-1} + \cdots + c_k$$

a polynomial of degree k with the leading coefficient of unity. There will also be other terms coming from the off-diagonal elements of $sI - A$ but none will have a degree as high as k. Thus we conclude that

$$|sI - A| = s^k + a_1 s^{k-1} + \cdots + a_k \qquad (3.51)$$

This is known as the *characteristic polynomial* of the matrix A. It plays a vital role in the dynamic behavior of the system. The roots of this polynomial are called the *characteristic roots*, or the *eigenvalues*, or the *poles*, of the system and determine the essential features of the unforced dynamic behavior of the system, since they determine the inverse Laplace transform of the resolvent, which is the transition matrix. See Chap. 4.

The adjoint of a k by k matrix is itself a k by k matrix whose elements are the cofactors of the original matrix. Each cofactor is obtained by computing the determinant of the matrix that remains when a row and a column of the original matrix are deleted. It thus follows that each element in adj $(sI - A)$ is a polynomial in s of maximum degree $k - 1$. (The polynomial cannot have degree k when any row and column of $sI - A$ is deleted.) Thus it is seen that the adjoint of $sI - A$ can be written

$$\text{adj }(sI - A) = E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k$$

Thus we can express the resolvent in the following form

$$(sI - A)^{-1} = \frac{E_1 s^{k-1} + \cdots + E_k}{s^k + a_1 s^{k-1} + \cdots + a_k} \qquad (3.52)$$

An interesting and useful relationship for the coefficient matrices E_i of the adjoint matrix can be obtained by multiplying both sides of (3.52) by $|sI - A|(sI - A)$. The result is

$$|sI - A|I = (sI - A)(E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k) \qquad (3.53)$$

or

$$s^k I + a_1 s^{k-1} I + \cdots + a_k I = s^k E_1 + s^{k-1}(E_2 - AE_1)$$
$$+ \cdots + s(E_k - AE_{k-1}) - AE_k$$

Equating the coefficients of s^i on both sides of (3.53) gives

$$E_1 = I$$
$$E_2 - AE_1 = a_1 I$$
$$E_3 - AE_2 = a_2 I$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \qquad (3.54)$$
$$E_k - AE_{k-1} = a_{k-1} I$$
$$-AE_k = a_k I$$

We have thus determined that the leading coefficient matrix of adj $(sI - A)$ is the identity matrix, and that the subsequent coefficients can be obtained recursively:

$$E_2 = AE_1 + a_1 I$$
$$E_3 = AE_2 + a_2 I$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \qquad (3.55)$$
$$E_k = AE_{k-1} + a_{k-1} I$$

The last equation in (3.54) is redundant, but can be used as a check, when the recursion equations (3.55) are used as the basis of a numerical algorithm. In this case the "check equation" can be written

$$E_{k+1} = AE_k + a_k I = 0 \qquad (3.56)$$

An algorithm based on (3.55) requires the coefficients a_i ($i = 1, \ldots, k$) of the characteristic polynomial. Fortunately, the determination of these coefficients can be included in the algorithm, for it can be shown that

$$a_1 = -\text{tr}(AE_1)$$
$$a_2 = -\tfrac{1}{2}\text{tr}(AE_2)$$

More generally

$$a_i = -\frac{1}{i}\text{tr}(AE_i) \qquad i = 1, 2, \ldots, k \qquad (3.57)$$
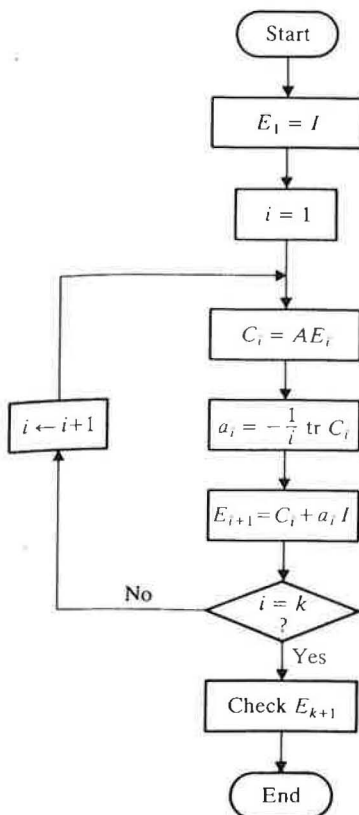
Figure 3.1 Algorithm for computing

$$(sI - A)^{-1} = \frac{E_1 s^{k-1} + \cdots + E_k}{s^k + a_1 s^{k-1} + \cdots + a_k}$$

An algorithm for computing the numerator matrices E_i and the coefficients a_i, starting with $E_1 = I$, is illustrated in the form of a flow chart in Fig. 3.1.

A proof of (3.57) is found in many textbooks such as [5, 6]. The algorithm based on (3.56) and (3.57) appears to have been discovered several times in various parts of the world. The names of Leverrier, Souriau, Faddeeva, and Frame are often associated with it.

This algorithm is convenient for hand calculation and easy to implement on a digital computer. Unfortunately, however, it is not a very good algorithm when the order k of the system is large (higher than about 10). The check matrix E_{k+1}, which is supposed to be zero, usually turns out to be embarrassingly large, and hence the resulting coefficients a_i and E_i are often suspect.

Example 3E Inertial navigation The equations for errors in an inertial navigation system are approximated by

$$\Delta \dot{x} = \Delta v$$

$$\Delta \dot{v} = -g \Delta \psi + E_A \qquad (3E.1)$$

$$\Delta \dot{\psi} = \frac{1}{R} \Delta v + E_G$$

where Δx is the position error, Δv is the velocity error, $\Delta \psi$ is the tilt of the platform, g is the acceleration of gravity, and R is the radius of the earth. (The driving terms are the accelerometer error E_A and the gyro error E_G.)

For the state variables defined by

$$x_1 = \Delta x \qquad x_2 = \Delta v \qquad x_3 = \Delta \psi$$

the A matrix is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & 1/R & 0 \end{bmatrix}$$

and, regarding E_A and E_G as inputs, the B matrix is

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrices appearing in the recursive algorithm are

$$C_1 = AE_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & 1/R & 0 \end{bmatrix} \qquad a_1 = -\text{tr } C_1 = 0 \qquad E_2 = C_1 + a_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -g \\ 0 & 1/R & 0 \end{bmatrix}$$

$$C_2 = AE_2 = \begin{bmatrix} 0 & 0 & -g \\ 0 & -g/R & 0 \\ 0 & 0 & -g/R \end{bmatrix} \quad \begin{aligned} a_2 &= -\tfrac{1}{2}(-2g/R) \\ &= g/R \end{aligned} \qquad E_3 = C_2 + a_2 I = \begin{bmatrix} g/R & 0 & -g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_3 = AE_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad a_3 = 0 \qquad E_4 = C_3 + a_3 I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus

$$(sI - A)^{-1} = \begin{bmatrix} s^2 + g/R & s & -g \\ 0 & s^2 & -gs \\ 0 & s/R & s^2 \end{bmatrix} \frac{1}{s^3 + (g/R)s}$$

$$= \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s^2 + g/R} & \dfrac{-g}{s(s^2 + g/R)} \\ 0 & \dfrac{s}{s^2 + g/R} & \dfrac{-g}{s^2 + g/R} \\ 0 & \dfrac{1/R}{s^2 + g/R} & \dfrac{s}{s^2 + g/R} \end{bmatrix} \qquad (3\text{E}.2)$$

The state transition matrix corresponding to the resolvent (3E.2) is obtained by taking its inverse Laplace transform.

$$\Phi(t) = \begin{bmatrix} 1 & \dfrac{\sin \Omega t}{\Omega} & \dfrac{g}{\Omega^2}(\cos \Omega t - 1) \\ 0 & \cos \Omega t & -\dfrac{g}{\Omega} \sin \Omega t \\ 0 & \dfrac{\sin \Omega t}{\Omega R} & \cos \Omega t \end{bmatrix} \qquad \Omega = \sqrt{g/R} \qquad (3\text{E}.3)$$

The elements of the state transition matrix, with the exception of ϕ_{11} are all oscillatory with a frequency $\Omega = \sqrt{g/R}$ which is the natural frequency of a pendulum of length equal to the earth's radius; $\Omega = 0.001\ 235$ rad/s corresponding to a period $T = 2\pi/\Omega = 84.4$ min., which is known as the "Schuler period." (See Note 3.4.)

Because the error equations are undamped, the effects of even small instrument biases can result in substantial navigation errors. Consider, for example, a constant gyro bias

$$E_G^{.} = \frac{c}{s}$$

The Laplace transform of the position error is given by

$$\Delta x(s) = \phi_{13}(s)\frac{c}{s} = -\frac{g}{s^2(s^2 + \Omega^2)}c \qquad (3E.4)$$

and the corresponding position error, as a function of time, is the inverse Laplace transform of (3E.4)

$$\Delta x(t) = -\frac{g}{\Omega^2}\left(t - \frac{1}{\Omega}\sin\Omega t\right)c \qquad (3E.5)$$

The position error consists of two terms: a periodic term at the Schuler period and a term which grows with time (also called a *secular term* at a rate of $-(g/\Omega^2)c = -Rc$. The position error thus grows at a rate proportional to the earth's radius. The position error will grow at a rate of about 70 m/h for each degree-per-hour "drift" ($E_G = c$) of the gyro.

3.5 INPUT-OUTPUT RELATIONS: TRANSFER FUNCTIONS

In conventional (frequency-domain) analysis of system dynamics attention is focused on the relationship between the output y and the input u. The focus shifts to the state vector when state space analysis is used, but there is still an interest in the input-output relation. Usually when an input-output analysis is made, the initial state $x(0)$ is assumed to be zero. In this case the Laplace transform of the state is given by

$$x(s) = (sI - A)^{-1}Bu(s) \qquad (3.58)$$

If the output is defined by

$$y(t) = Cx(t) \qquad (3.59)$$

Then its Laplace transform is

$$y(s) = Cx(s) \qquad (3.60)$$

and, by (3.58)

$$y(s) = C(sI - A)^{-1}Bu(s) \qquad (3.61)$$

The matrix

$$H(s) = C(sI - A)^{-1}B \qquad (3.62)$$

that relates the Laplace transform of the output to the Laplace transform of the input is known as the *transfer-function* matrix.