Time Domain **Analysis** of
Continuous Time Systems

ELEC 3004: **Systems**: Signals & Controls

Dr. Surya Singh

Lecture 7

elec3004@itee.uq.edu.au
http://robotics.itee.uq.edu.au/~elec3004/

March 20, 2013

Announcements:

- Assignment 1 is up on Platypus!
  - Due at the end of next week!
  - Next week's tutorial sessions are Q&A meetings

Lab 1:

- Don't forget the Pre-Lab
- Extra Session Needed ????

- Space Audit this **Friday** and next **Wednesday**
  - Sounds like an ideal time for a **pop-quiz**

Today:

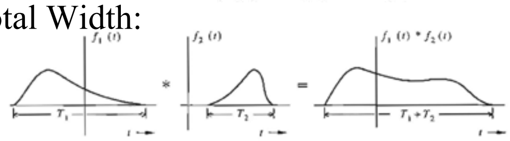
Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Signals & Signal Models
	8-Mar	System Models
3	13-Mar	Linear Dynamical Systems
	15-Mar	Sampling & Data Acquisition
4	20-Mar	Time Domain Analysis of Continuous Time Systems
	22-Mar	System Behaviour & Stability
5	27-Mar	Signal Representation
	29-Mar	Holiday
6	10-Apr	Frequency Response & Fourier Transform
	12-Apr	Analog Filters
7	17-Apr	IIR Systems
	19-Apr	FIR Systems
8	24-Apr	z-Transform
	26-Apr	Discrete-Time Signals
9	1-May	Discrete-Time Systems
	3-May	Digital Filters
10	8-May	State-Space
	10-May	Controllability & Observability
11	15-May	Introduction to Digital Control
	17-May	Stability of Digital Systems
12	22-May	PID & Computer Control
	24-May	Information Theory & Communications
13	29-May	Applications in Industry
	31-May	Summary and Course Review

Convolution & Properties

$$f_1(t) * f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

Properties:

- Commutative: $f_1(t)*f_2(t) = f_2(t)*f_1(t)$
- Distributive: $f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$
- Associative: $f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t)$
- Shift:
  if $f_1(t)*f_2(t)=c(t)$, then $f_1(t-\mathbf{T})*f_2(t)= f_1(t)*f_2(t-\mathbf{T})=c(t-\mathbf{T})$
- Identity (Convolution with an Impulse):
  $f(t) * \delta(t) = f(t)$
- Total Width:



Based on Lathi, *SPLS*, Sec 2.4-1

Convolution & Properties [II]

- Convolution systems are **linear**:

$$h * (\alpha u_1 + \beta u_2) = \alpha(h * u_1) + \beta(h * u_2)$$

- Convolution systems are **causal**: the output y(t) at time t depends only on past inputs

- Convolution systems are **time-invariant** (if we shift the signal, the output similarly shifts)

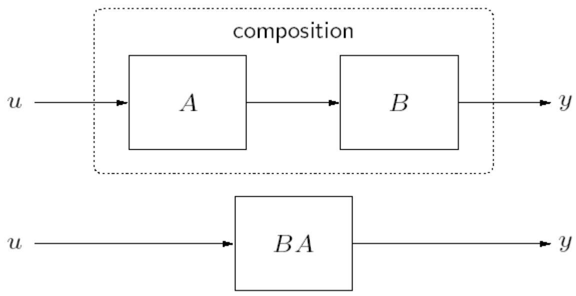
➔

$$\tilde{u}(t) = \begin{cases} 0 & t < T \\ u(t-T) & t \geq 0 \end{cases}$$

$$\tilde{y}(t) = \begin{cases} 0 & t < T \\ y(t-T) & t \geq 0 \end{cases}$$

Convolution & Properties [III]

- Composition of convolution systems corresponds to:
  - multiplication of transfer functions
  - convolution of impulse responses



- Thus:
  - We can manipulate block diagrams with transfer functions as if they were simple gains
  - convolution systems commute with each other

Convolution & Systems

- Convolution system with input u (u(t) = 0, t < 0) and output y:

$$y(t) = \int_0^t h(\tau)u(t-\tau)\, d\tau = \int_0^t h(t-\tau)u(\tau)\, d\tau$$

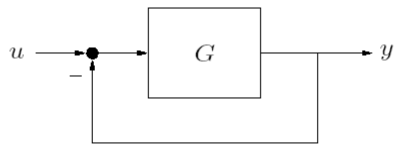
- abbreviated:

$$y = h * u$$

- in the frequency domain:

$$Y(s) = H(s)U(s)$$

Convolution & Feedback



- In the time domain:

$$y(t) = \int_0^t g(t-\tau)(u(\tau) - y(\tau))\, d\tau$$

- In the frequency domain:
  - Y=G(U-Y)

  ➔Y(s) = H(s)U(s)

$$H(s) = \frac{G(s)}{1+G(s)}$$

Graphical Understanding of Convolution

➔ For $c(\tau) = (f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$:

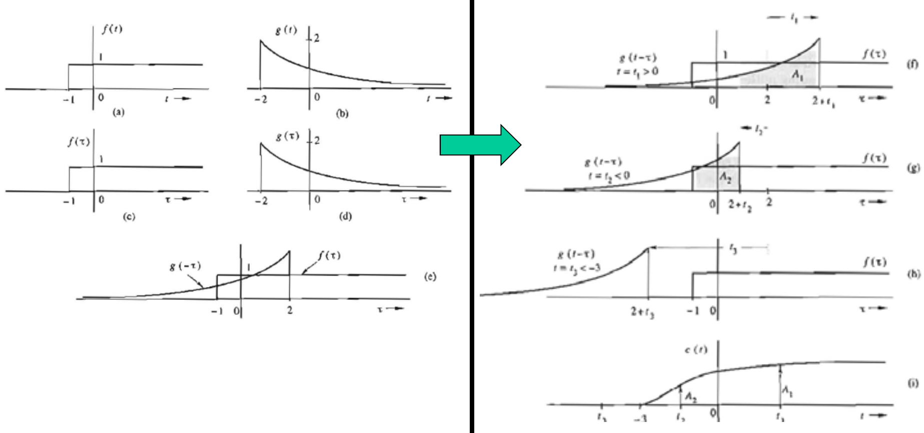
1. Keep the function $f(\tau)$ fixed
2. **Flip** (invert) the function $g(\tau)$ about the vertical axis (τ=0)
   = this is $g(-\tau)$
3. **Shift** this frame ($g(-\tau)$) along τ (horizontal axis) by t_0.
   = this is $g(t_0 - \tau)$

➔ For $c(t_0)$:

4. $c(t_0)$ = the area under the product of $f(\tau)$ **and** $g(t_0 - \tau)$

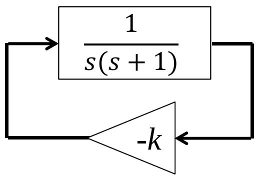
5. Repeat this procedure, shifting the frame by different values (positive and negative) to obtain **c(t) for all values of t**.

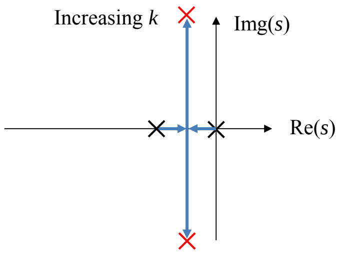
Graphical Understanding of Convolution (Ex)



Recall the **Root Locus**

- We know that under feedback gain, the poles of the closed-loop system move
  - The root locus tells us where they go!
  - We can solve for this analytically*





  - Root loci can be plotted for all sorts of parameters, not just gain!

The Root Locus

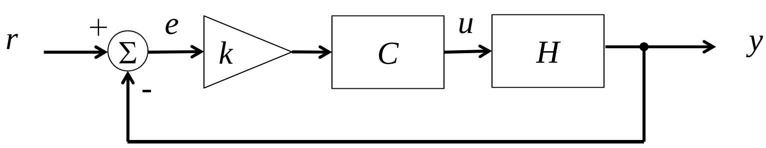
- We often care about the effect of increasing gain of a control compensator design:

$$\frac{y}{r} = \frac{kCH}{1 + kCH}$$

Multiplying by denominator:

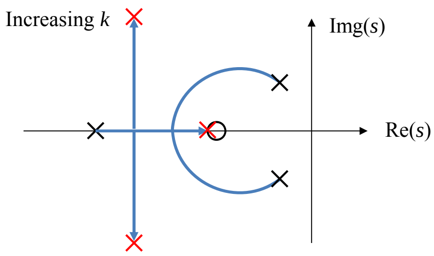
$$\frac{y}{r} = \frac{kC_nH_n}{C_dH_d + kCnHn}$$

characteristic polynomial



The root locus

- Pole positions change with increasing gain
  - The trajectory of poles on the pole-zero plot with changing k is called the "root locus"
  - This is sometimes quite complex



  - (In practice you'd plot these with computers)

Root Locus Drawing Rules

1. The root locus is symmetric with respect to the real axis.

2. The root loci start from n poles p_i (when $K = 0$) and approach the n zeros (m finite zeros z_i and $n - m$ infinite zeros when $K \to \infty$).

3. The root locus includes all points on the real axis to the left of an odd number of open-loop real poles and zeros.

4. As $K \to \infty$, $n - m$ branches of the root-locus approach asymptotically $n - m$ straight lines (called **asymptotes**) with angles

$$\theta = \frac{(2k+1)180^\circ}{n-m}, \quad k = 0, \pm 1, \pm 2, \ldots$$

and the starting point of all asymptotes is on the real axis at

$$\kappa = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m} = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}.$$

Adapted from Qui, *IFC* 2010 – p. 201

Root Locus Drawing Rules [II]

5. The **breakaway points** (where the root loci meet and split away, usually on real axis) and the **breakin points** (where the root loci meet and enter the real axis) are among the roots of the equation: $\frac{dL(s)}{ds} = 0$. (On the real axis, only those roots that satisfy Rule 3 are breakaway or breakin points.)
6. The **departure angle** ϕ_k (from a pole, p_k) is given by

$$\phi_k = \sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{j=1, j\neq k}^{n} \angle(p_k - p_j) \pm 180^\circ.$$

(In the case p_k is l repeated poles, the departure angle becomes ϕ_k/ℓ.)
The **arrival angle** ψ_k (at a zero, z_k) is given by

$$\psi_k = -\sum_{i=1, i\neq k}^{m} \angle(z_k - z_i) + \sum_{j=1}^{n} \angle(z_k - p_j) \pm 180^\circ.$$

(In the case z_k is l repeated zeros, the arrival angle becomes ψ_k/ℓ.)

TABLE 5.1: Root locus rules: $0 \le K \le \infty$.

Adapted from Qui, *IFC* 2010 – p. 201

Recall (Lecture 5, Slide 10): Linear Dynamic [Differential] System

≡ LTI systems for which the input & output are linear ODEs

$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

Laplace:

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s)Y(s) = B(s)X(s)$$

- Total response = Zero-input response + Zero-state response

  Zero-input response: **Initial conditions**; Zero-state response: **External Input**

Recall: Second Order Systems

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a(\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}(y'')}) + b(\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}) + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

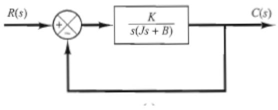
Second Order Systems

so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

Second Order Response



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

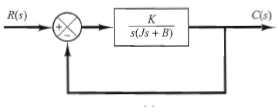
Three Types:

- I: Underdamped: $(0 < \zeta < 1)$:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)} \qquad \omega_d = \omega_n\sqrt{1 - \zeta^2}.$$

$$\begin{aligned}\mathcal{L}^{-1}[C(s)] &= c(t)\\ &= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right),\end{aligned}$$

Second Order Response



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

Three Types:

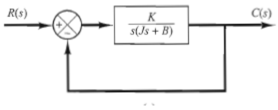
- II: Critically Damped: $(\zeta = 1)$:

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

$$\lim_{\zeta \to 1} \frac{\sin\omega_d t}{\sqrt{1-\zeta^2}} = \lim_{\zeta \to 1} \frac{\sin\omega_n\sqrt{1-\zeta^2}\,t}{\sqrt{1-\zeta^2}} = \omega_n t$$

Second Order Response



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

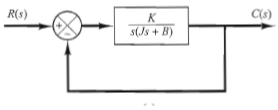
Three Types:

- III: Over Damped: $(\zeta > 1)$:

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

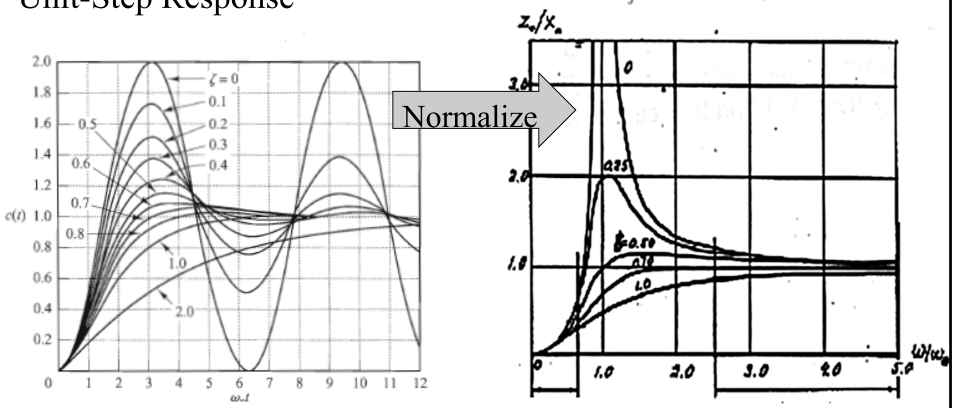
Second Order Response



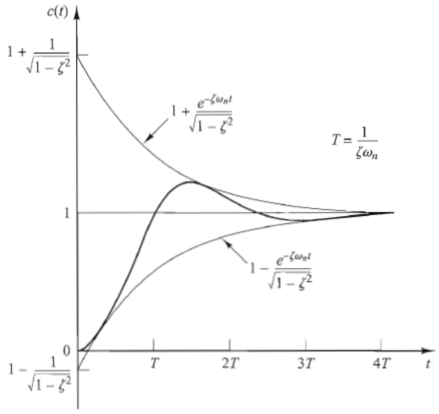
$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

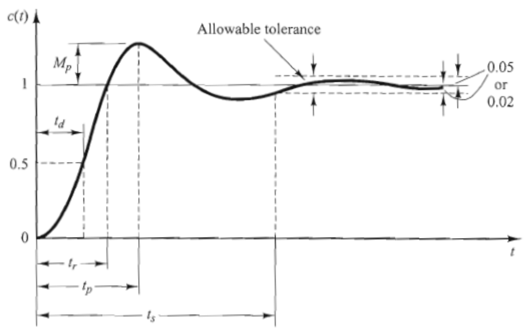
Unit-Step Response



Second Order Response
Envelope Curves



Second Order Response
Unit Step Response Terms



- Delay time, t_d: The time required for the response to reach half the final value
- Rise time, t_r: The time required for the response to rise from 10% to 90%
- Peak time, t_p:The time required for the response to reach the first peak of the overshoot
- Maximum (percent) overshoot, Mp:

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

- Settling time, t_s: The time to be within 2-5% of the final value

Second Order Response
Seeing this on the S-plane

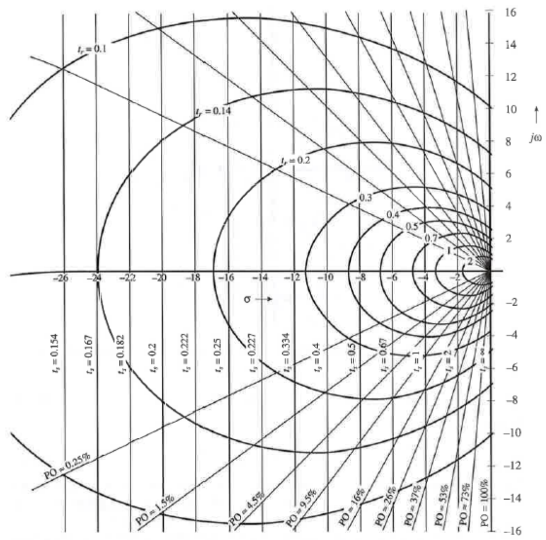
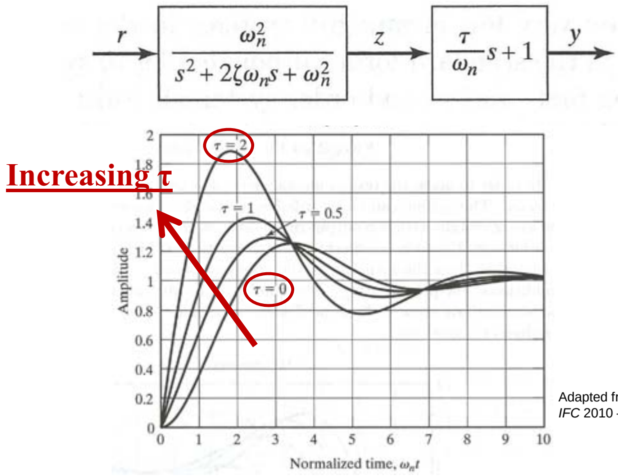


Fig. 6.40 Contours of second-order system pole location for constant PO, constant t_s, and constant t_r in s plane.

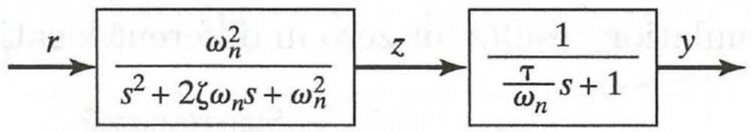
Second Order Response
The Case of **Adding a Zero**

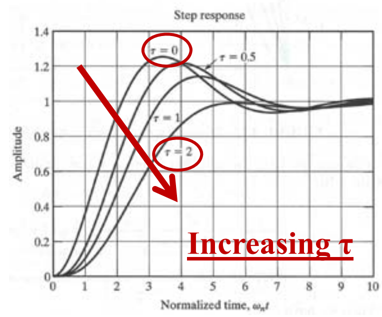


Adapted from Qui, *IFC* 2010 – pp. 154-5

- The addition of a zero (a **s** term) gives a system with a shorter rise time, a shorter peak time, and a larger overshoot

Second Order Response
The Case of **Adding a Zero**

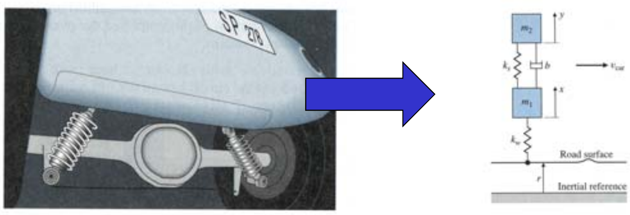


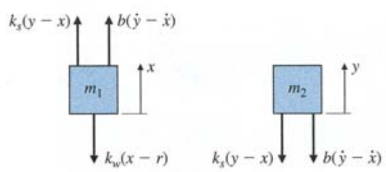


- The addition of a pole (a **1/s** term) **slows down** the system response and reduces the overshoot.

Adapted from Qui, *IFC* 2010 – pp. 154-5

Example: Quarter-Car Model





Example: Quarter-Car Model (2)

$$\ddot{x} + \frac{b}{m_1}(\dot{x} - \dot{y}) + \frac{k_s}{m_1}(x - y) + \frac{k_w}{m_1}x = \frac{k_w}{m_1}r,$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y} - \dot{x}) + \frac{k_s}{m_2}(y - x) = 0.$$

$$s^2X(s) + s\frac{b}{m_1}(X(s) - Y(s)) + \frac{k_s}{m_1}(X(s) - Y(s)) + \frac{k_w}{m_1}X(s) = \frac{k_w}{m_1}R(s),$$

$$s^2Y(s) + s\frac{b}{m_2}(Y(s) - X(s)) + \frac{k_s}{m_2}(Y(s) - X(s)) = 0,$$

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}.$$

Next Time…

- Stability
  - A performance measure which informs the extent to which all the all the poles of the transfer function have negative real parts
  - Aka:
  - Attempts to spontaneously disassemble itself

or 

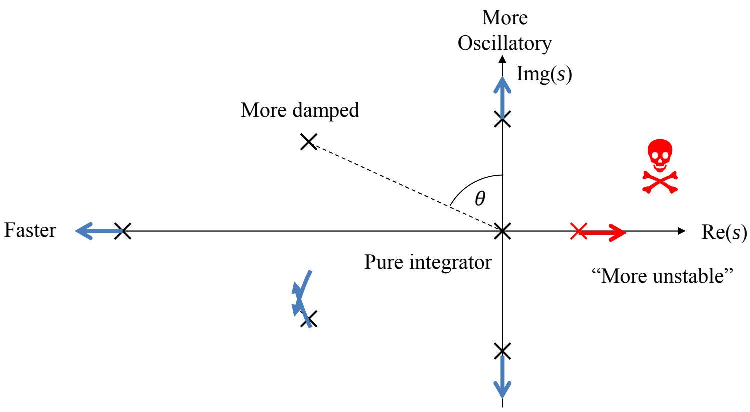
- Review:
  - Section 3.10 of Lathi

“Back to the Future”: Laplace Review!

Source: *Wikipedia*, Laplace

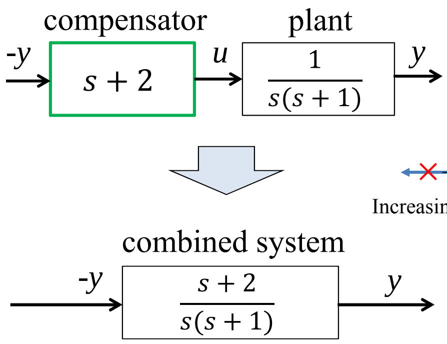
Recall dynamic responses

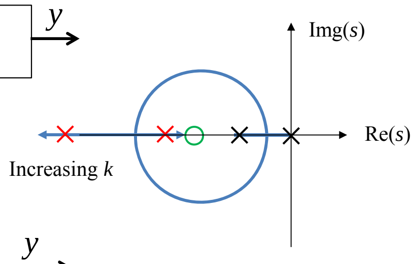
- Moving pole positions change system response characteristics



Dynamic compensation

- We can do more than just apply gain!
  - We can add dynamics into the controller that alter the open-loop response





But what dynamics to add?

- Recognise the following:
  - A root locus starts at poles, terminates at zeros
  - "Holes eat poles"
  - Closely matched pole and zero dynamics cancel
  - The locus is on the real axis to the left of an odd number of poles (treat zeros as 'negative' poles)