Linear Dynamical Systems

ELEC 3004: **Systems**: Signals & Controls

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Lecture 5

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Announcements:

- Assignment 1 is coming!

- Space Audit on **March 22** (F) and **March 28** (W)
  - P&F is going around in **black hats**
  - Sounds like an ideal time for a **pop-quiz**
  - Bring a friend or two (just saying ☺)

Today:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Signals & Signal Models
	8-Mar	System Models
3	13-Mar	Linear Dynamical Systems
	15-Mar	Sampling & Data Acquisition
4	20-Mar	Time Domain Analysis of Continuous Time Systems
	22-Mar	System Behaviour & Stability
5	27-Mar	Signal Representation
	29-Mar	Holiday
6	10-Apr	Frequency Response & Fourier Transform
	12-Apr	Analog Filters
7	17-Apr	IIR Systems
	19-Apr	FIR Systems
8	24-Apr	z-Transform
	26-Apr	Discrete-Time Signals
9	1-May	Discrete-Time Systems
	3-May	Digital Filters
10	8-May	State-Space
	10-May	Controllability & Observability
11	15-May	Introduction to Digital Control
	17-May	Stability of Digital Systems
12	22-May	PID & Computer Control
	24-May	Information Theory & Communications
13	29-May	Applications in Industry
	31-May	Summary and Course Review

Recall From Last Time …
Linear Systems: **The Concept Of Linearity**

- Additvitiy

$$x_1 \longrightarrow y_1 \quad \text{and} \quad x_2 \longrightarrow y_2$$

$$x_1 + x_2 \longrightarrow y_1 + y_2$$

- Homogeneity (scaling)

$$kx \longrightarrow ky$$

- Superposition

$$k_1x_1 + k_2x_2 \longrightarrow k_1y_1 + k_2y_2$$

Recall From Last Time …
Classifications of Systems

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Causality:
Causal (physical or nonanticipative) systems

- Is one for which the output at any instant t_0 depends only on the value of the input x(t) for $t \leq t_0$. Ex:

$u(t) = x(t-2) \Rightarrow \text{causal}$ $u(t) = x(t-2) + x(t+2) \Rightarrow \text{noncausal}$

- A "*real-time*" system must be causals
  - How can it respond to future inputs?
- A prophetic system: knows future inputs and acts on it (now)
  - The output would begin before t_0
- In some cases Noncausal maybe modelled as causal with delay
- Noncausal systems provide **an upper bound** on the performance of causal systems

Causality:
Looking at this from the output's perspective…

- **Causal** = The output *before* some time *t* does not depend on the input *after* time *t.*

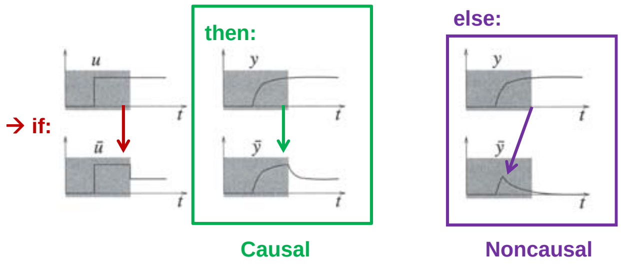
Given: $y(t) = F(u(t))$

For:

$\hat{u}(t) = u(t)\,,\ \forall 0 \le t < T\ or\ [0, T)$

Then for a T>0:

$\rightarrow \hat{y}(t) = y(t)\,,\ \forall 0 \le t < T$

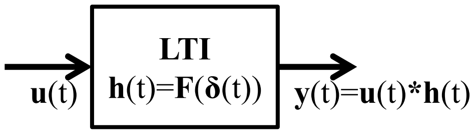


Dynamical Systems…

- A system with a memory
  - Where past history (or derivative states) are **relevant** in determining the response
- Ex:
  - RC circuit: Dynamical
    - Clearly a function of the "capacitor's past" (initial state) and
    - Time! (charge / discharge)
  - R circuit: is memoryless ∵ the output of the system (recall V=IR) at some time **t** only depends on the input at time **t**

- Lumped/Distributed
  - Lumped: Parameter is constant through the process & can be treated as a "point" in space
- Distributed: System dimensions ≠ small over signal
  - Ex: waveguides, antennas, microwave tubes, etc.

Linear Time Invariant



- Linear & Time-invariant (of course - tautology!)
- Impulse response: **h**(t)=**F**(**δ**(t))
- Why?
  - Since it is linear the output response (**y**) to any input (**x**) is:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$
$$y(t) = F\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right] \overset{linear}{\rightarrow} \int_{-\infty}^{\infty} x(\tau)\,F\left[\delta(t-\tau)\right]d\tau$$
$$h(t-\tau) \overset{TI}{\equiv} F\left[\delta(t-\tau)\right]$$
$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

- The output of any continuous-time LTI system is the **convolution** of input **u**(t) with the impulse response **F**(**δ**(t)) of the system.

Linear Dynamic [Differential] System

≡ LTI systems for which the input & output are linear ODEs

$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

Laplace:

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s) Y(s) = B(s) X(s)$$

- Total response = Zero-input response + Zero-state response

Initial conditions (Zero-input response) **External Input** (Zero-state response)

Linear Systems and ODE's

- Linear system described by differential equation

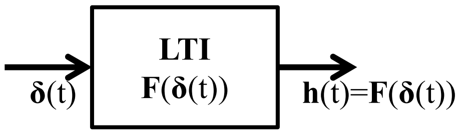
$$a_0 y \quad a_1 \frac{dy}{dt} \quad \cdots \quad a_n \frac{d^n y}{dt^n} \quad b_0 x \quad b_1 \frac{dx}{dt} \quad \cdots \quad b_m \frac{d^m x}{dt^m}$$

- Which using Laplace Transforms can be written as

$$a_0 Y(s) + a_1 sY(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 sX(s) + \cdots + b_m s^m X(s)$$
$$A(s)Y(s) = B(s)X(s)$$

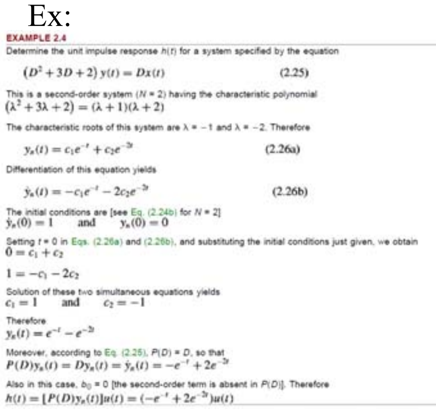
where *A(s)* and *B(s)* are polynomials in *s*

Unit Impulse Response



- $\delta(t)$: Impulsive excitation
- h(t): characteristic mode terms

Ex:



EXAMPLE 2.4

Determine the unit impulse response $h(t)$ for a system specified by the equation

$$(D^2 + 3D + 2)\, y(t) = Dx(t) \qquad (2.25)$$

This is a second-order system ($N = 2$) having the characteristic polynomial
$(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2)$

The characteristic roots of this system are $\lambda = -1$ and $\lambda = -2$. Therefore

$$y_n(t) = c_1 e^{-t} + c_2 e^{-2t} \qquad (2.26a)$$

Differentiation of this equation yields

$$\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \qquad (2.26b)$$

The initial conditions are [see Eq. (2.24b) for $N = 2$]
$\dot{y}_n(0) = 1$ and $y_n(0) = 0$

Setting $t = 0$ in Eqs. (2.26a) and (2.26b), and substituting the initial conditions just given, we obtain
$0 = c_1 + c_2$

$1 = -c_1 - 2c_2$

Solution of these two simultaneous equations yields
$c_1 = 1$ and $c_2 = -1$

Therefore
$y_n(t) = e^{-t} - e^{-2t}$

Moreover, according to Eq. (2.25), $P(D) = D$, so that
$P(D)y_n(t) = Dy_n(t) = \dot{y}_n(t) = -e^{-t} + 2e^{-2t}$

Also in this case, $b_0 = 0$ [the second-order term is absent in $P(D)$]. Therefore
$h(t) = [P(D)y_n(t)]u(t) = (-e^{-t} + 2e^{-2t})u(t)$

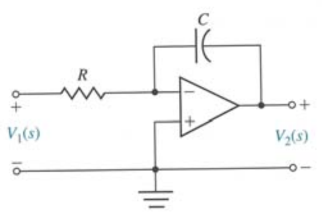
System Models

- Various things – all the same!

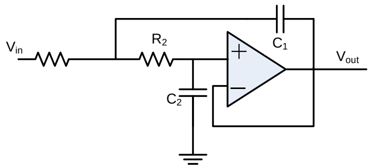
Table 2.1 Summary of Through- and Across-Variables for Physical Systems

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Circuits



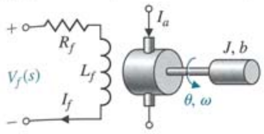
$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$



$$\frac{v_{out}}{v_{in}} = \frac{1}{C_1C_2R_1R_2s^2 + C_2(R_1+R_2)s + 1}$$

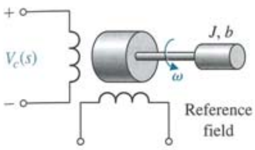
Motors

5. DC motor, field-controlled, rotational actuator



$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$$

7. AC motor, two-phase control field, rotational actuator



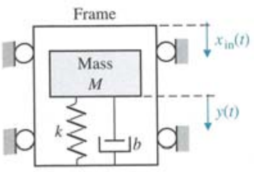
$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$$

$$\tau = J/(b - m)$$

m = slope of linearized torque-speed curve (normally negative)

Mechanical Systems

15. Accelerometer, acceleration sensor



$$x_o(t) = y(t) - x_{in}(t),$$

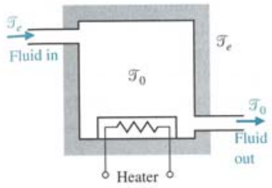
$$\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$$

For low-frequency oscillations, where $\omega < \omega_n$,

$$\frac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \frac{\omega^2}{k/M}$$

Thermal Systems

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e$ = temperature difference due to thermal process

C_t = thermal capacitance

Q = fluid flow rate = constant

S = specific heat of water

R_t = thermal resistance of insulation

$q(s)$ = transform of rate of heat flow of heating element

First Order Systems

First order systems

$$ay' + by = 0 \quad (\text{with } a \neq 0)$$

righthand side is zero:

- called *autonomous system*
- solution is called *natural* or *unforced response*

can be expressed as

$$Ty' + y = 0 \quad \text{or} \quad y' + ry = 0$$

where

- $T = a/b$ is a *time* (units: seconds)
- $r = b/a = 1/T$ is a *rate* (units: 1/sec)

First Order Systems

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T(\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}) + Y(s) = 0$$

solve for $Y(s)$ (algebra!)

$$Y(s) = \frac{Ty(0)}{sT + 1} = \frac{y(0)}{s + 1/T}$$

and so $y(t) = y(0)e^{-t/T}$

First Order Systems

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, y *decays* exponentially

- T gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half $(0.693 = \log 2)$
- $4.6T$ gives time to decay by 0.01 $(4.6 = \log 100)$

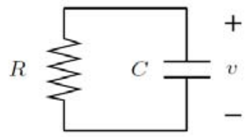
if $T < 0$, y *grows* exponentially

- $|T|$ gives time to grow by $e \approx 2.72$;
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100

First Order Systems

Examples

simple RC circuit:



circuit equation: $RCv' + v = 0$

solution: $v(t) = v(0)e^{-t/(RC)}$

population dynamics:

- $y(t)$ is population of some bacteria at time t
- growth (or decay if negative) rate is $y' = by - dy$ where b is birth rate, d is death rate
- $y(t) = y(0)e^{(b-d)t}$ (grows if $b > d$; decays if $b < d$)

Second Order Systems

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a\underbrace{(s^2Y(s) - sy(0) - y'(0))}_{\mathcal{L}(y'')} + b\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

Second Order Systems

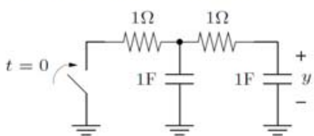
so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

Ex: RC Circuit

Example: second-order RC circuit



at $t = 0$, the voltage across each capacitor is 1V

- for $t \geq 0$, y satisfies LCCODE (from page 2-18)

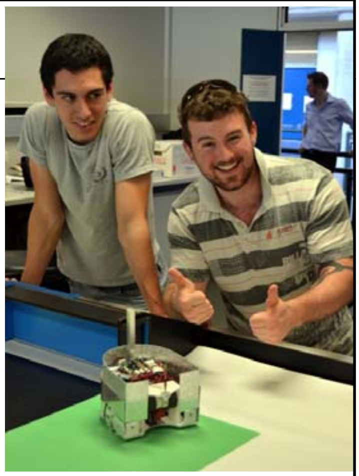
$$y'' + 3y' + y = 0$$

- initial conditions:

$$y(0) = 1, \quad y'(0) = 0$$

(at $t = 0$, voltage across righthand capacitor is one; current through righthand resistor is zero)

Optical Flow Experiments



- Adam is still looking for volunteers
- For those signed up
  - **Please** reply to his email
- Additional times are available:
  - Thursday 3p-5p
  - Friday 10a-12n

*Come Go With The **Flow!***

Next Time…

- Sampling
  - Measurements at regular intervals of a continuous signal
  - Not to be confused with
    " How to try regional dishes without indigestion"
- Review:
  - Chapter 8 of Lathi
- Send (and you shall receive) a positive signal ☺