State-Space:
Controllably & Observability

ELEC 3004: **Systems**: Signals & Controls

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(with material from Dr. Paul Pounds)

Lecture 23

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May 23, 2013

Today in Linear Systems…

1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Signals & Signal Models
	8-Mar	System Models
3	13-Mar	Linear Dynamical Systems
	15-Mar	Sampling & Data Acquisition
4	20-Mar	Time Domain Analysis of Continuous Time Systems
	22-Mar	System Behaviour & Stability
5	27-Mar	Signal Representation
	29-Mar	Holiday
6	10-Apr	Frequency Response
	12-Apr	z-Transform
7	17-Apr	Noise & Filtering
	19-Apr	Analog Filters
8	24-Apr	Discrete-Time Signals
	26-Apr	Discrete-Time Systems
9	1-May	Digital Filters & IIR/FIR Systems
	3-May	Fourier Transform & DTFT
10	8-May	Introduction to Digital Control
	10-May	Stability of Digital Systems
11	15-May	PID & Computer Control
	17-May	Applications in Industry
12	22-May	State-Space
	24-May	Controllability & Observability
13	29-May	Information Theory/Communications & Review
	31-May	Summary and Course Review

Goals for the Week

Today:

- State-Space
- Compensator Design

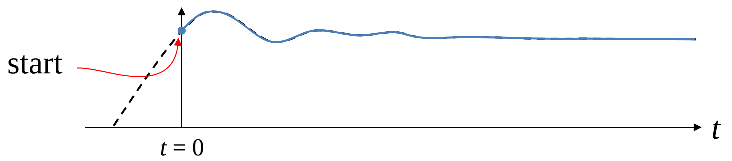
Friday:

- Controllability
- Observability

Affairs of state

- Introductory brain-teaser:
  - If you have a dynamic system model with history (ie. integration) how do you represent the instantaneous state of the plant?

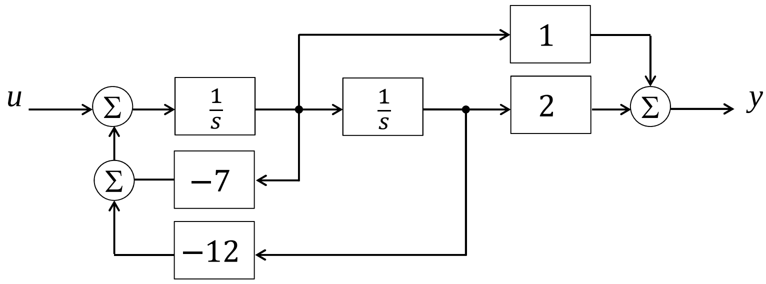
Eg. how would you setup a simulation of a step response, mid-step?



Introduction to state-space

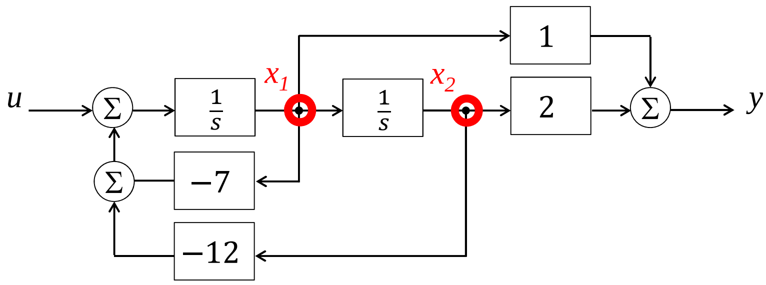
- Linear systems can be written as networks of simple dynamic elements:

$$H = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3}$$



Introduction to state-space

- We can identify the nodes in the system
  - These nodes contain the integrated time-history values of the system response
  - We call them "states"

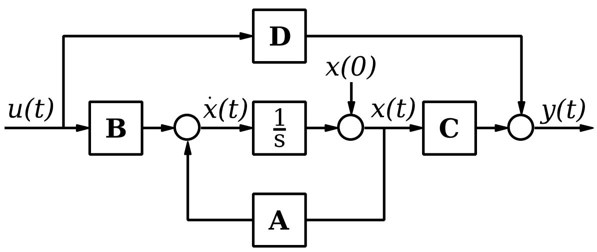


State-Space Terminology

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

- **u**: Input: $u : [0, \infty) \to \mathbb{R}^k$
- **x**: State: $x : [0, \infty) \to \mathbb{R}^n$
- **y**: Output $y : [0, \infty) \to \mathbb{R}^m$

State-Space Terminology



$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

LTI State-Space

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

- **If** the system is **linear and time invariant**,
  then A,B,C,D are constant coefficient

$$\rightarrow \dot{x} = Ax + Bu$$
$$\rightarrow y = Cx + Du$$

Discrete Time State-Space

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

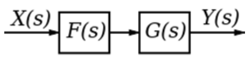
- **If** the system is **discrete**,
  then x and u are given by difference equations

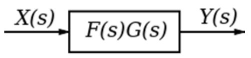
$$\rightarrow x[k+1] = A[k]\,x[k] + B[k]\,u[k]$$
$$y[k] = C[k]\,x[k] + D[k]\,u[k]$$

$$\rightarrow x^{+} = Ax + Bu$$
$$y = Cx + Du$$

Block Diagram Algebra in State Space

- Series:





System 1:

$$x'_F = A_F x_F + B_F u$$
$$y_F = C_F x_F + D_F u$$

System 2:

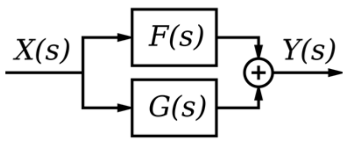
$$x'_G = A_G x_G + B_G y_F$$
$$y_G = C_G x_G + D_G y_F$$

$$\begin{bmatrix} x'_G \\ x'_F \end{bmatrix} = \begin{bmatrix} A_G & B_G C_F \\ 0 & A_F \end{bmatrix} \begin{bmatrix} x_G \\ x_F \end{bmatrix} + \begin{bmatrix} B_G D_F \\ B_F \end{bmatrix} u$$

$$\begin{bmatrix} y_G \\ y_F \end{bmatrix} = \begin{bmatrix} C_G & D_G C_F \\ 0 & C_F \end{bmatrix} \begin{bmatrix} x_G \\ x_F \end{bmatrix} + \begin{bmatrix} D_G D_F \\ D_F \end{bmatrix} u$$

Block Diagram Algebra in State Space

- Parallel:



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_1 + D_2) u$$

State-space representation

- State-space matrices are not necessarily a unique representation of a system
  - There are two common forms
- Control canonical form
  - Each node – each entry in $\boldsymbol{x}$ – represents a state of the system (each order of s maps to a state)
- Modal form
  - Diagonals of the state matrix $\mathbf{A}$ are the poles (“modes”) of the transfer function

Control canonical form

- CCF matrix representations have the following structure:

$$\begin{bmatrix} -a_1 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 & 0 \\ 0 & 1 & & & & \\ \vdots & & \ddots & & & \vdots \\ & & & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

Pretty diagonal!

State variable transformation

- Important note!
  - The states of a control canonical form system are not the same as the modal states
  - They represent the same dynamics, and give the same output, but the vector values are different!

- However we can convert between them:
  - Consider state representations, $\boldsymbol{x}$ and $\boldsymbol{q}$ where

$$\boldsymbol{x} = \mathbf{T}\boldsymbol{q}$$

T is a "transformation matrix"

State variable transformation

- Two homologous representations:

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}u \qquad \text{and} \qquad \dot{\boldsymbol{q}} = \mathbf{F}\boldsymbol{q} + \mathbf{G}u$$
$$y = \mathbf{C}\boldsymbol{x} + Du \qquad\qquad\qquad y = \mathbf{H}\boldsymbol{q} + Ju$$

We can write:

$$\dot{\boldsymbol{x}} = \mathbf{T}\dot{\boldsymbol{q}} = \mathbf{AT}\boldsymbol{z} + \mathbf{B}u$$
$$\dot{\boldsymbol{q}} = \mathbf{T}^{-1}\mathbf{AT}\boldsymbol{z} + \mathbf{T}^{-1}\mathbf{B}u$$

Therefore, $\mathbf{F} = \mathbf{T}^{-1}\mathbf{AT}$ and $\mathbf{G} = \mathbf{T}^{-1}\mathbf{B}$

Similarly, $\mathbf{C} = \mathbf{HT}$ and $D = J$

Controllability matrix

- To convert an arbitrary state representation in **F**, **G**, **H** and *J* to control canonical form **A**, **B**, **C** and *D*, the "controllability matrix"

$$\mathcal{C} = [\mathbf{G} \quad \mathbf{FG} \quad \mathbf{F}^2\mathbf{G} \quad \cdots \quad \mathbf{F}^{n-1}\mathbf{G}]$$

must be nonsingular.

Why is it called the "controllability" matrix?

Controllability matrix

- If you can write it in CCF, then the system equations must be linearly independent.
- Transformation by any nonsingular matrix preserves the controllability of the system.
- Thus, a nonsingular controllability matrix means ***x*** can be driven to any value.

Why is this "Kind of awesome"?

- The controllability of a system depends on the particular set of states you chose
- You can't tell just from a transfer function whether all the states of $\boldsymbol{x}$ are controllable
- The poles of the system are the Eigenvalues of **F**, (p_i).

State evolution

- Consider the system matrix relation:

$$\dot{\boldsymbol{x}} = \mathbf{F}\boldsymbol{x} + \mathbf{G}u$$
$$y = \mathbf{H}\boldsymbol{x} + Ju$$

The time solution of this system is:

$$\boldsymbol{x}(t) = e^{\mathbf{F}(t-t_0)}\,\boldsymbol{x}(t_0) + \int_{t_0}^{t} = e^{\mathbf{F}(t-\tau)}\,\mathbf{G}u(\tau)d\tau$$

If you didn't know, the matrix exponential is:

$$e^{\mathbf{K}t} = \mathbf{I} + \mathbf{K}t + \frac{1}{2!}\mathbf{K}^2t^2 + \frac{1}{3!}\mathbf{K}^3t^3 + \cdots$$

Stability

- We can solve for the natural response to initial conditions $\boldsymbol{x_0}$:

$$\boldsymbol{x}(t) = e^{p_i t}\boldsymbol{x}_0$$
$$\therefore \dot{\boldsymbol{x}}(t) = p_i e^{p_i t}\boldsymbol{x}_0 = \mathbf{F}e^{p_i t}\boldsymbol{x}_0$$

Clearly, a system will be stable provided
$\text{eig}(\mathbf{F}) < 0$

Characteristic polynomial

- From this, we can see $\mathbf{F}\boldsymbol{x}_0 = p_i\boldsymbol{x}_0$

or, $(p_i\mathrm{I} - \mathbf{F})\boldsymbol{x}_0 = 0$

which is true only when $\det(p_i\mathrm{I} - \mathbf{F})\boldsymbol{x}_0 = 0$

Aka. the characteristic equation!

- We can reconstruct the CP in s by writing:

$$\det(s\mathrm{I} - \mathbf{F})\boldsymbol{x}_0 = 0$$

Great, so how about control?

- Given $\dot{\boldsymbol{x}} = \mathbf{F}\boldsymbol{x} + \mathbf{G}u$, if we know $\mathbf{F}$ and $\mathbf{G}$, we can design a controller $u = -\mathbf{K}\boldsymbol{x}$ such that

$$\text{eig}(\mathbf{F} - \mathbf{GK}) < 0$$

- In fact, if we have full measurement and control of the states of $\boldsymbol{x}$, we can position the poles of the system in arbitrary locations!

(Of course, that never happens in reality.)

Example: PID control

- Consider a system parameterised by three states:
  - x_1, x_2, x_3
  - where $x_2 = \dot{x}_1$ and $x_3 = \dot{x}_2$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix} \boldsymbol{x} - \mathbf{K}u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + 0u$$

x_2 is the output state of the system;
x_1 is the value of the integral;
x_3 is the velocity.

- We can choose **K** to move the eigenvalues of the system as desired:

$$\det\begin{bmatrix}1 - K_1 & & \\ & 1 - K_2 & \\ & & -2 - K_3\end{bmatrix} = \mathbf{0}$$

All of these eigenvalues must be positive.

It's straightforward to see how adding derivative gain K_3 can stabilise the system.

Just scratching the surface

- There is a lot of stuff to state-space control
- One lecture (or even two) can't possibly cover it all in depth

Go play with Matlab and check it out!

Discretisation FTW!

- We can use the time-domain representation to produce difference equations!

$$\boldsymbol{x}(kT+T) = e^{\mathbf{F}T}\ \boldsymbol{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

Notice $\boldsymbol{u}(\tau)$ is not based on a discrete ZOH input, but rather an integrated time-series.

We can structure this by using the form:

$$u(\tau) = u(kT), \qquad kT \le \tau \le kT+T$$

Discretisation FTW!

- Put this in the form of a new variable:

$$\eta = kT + T - \tau$$

Then:

$$\boldsymbol{x}(kT+T) = e^{\boldsymbol{F}T}\boldsymbol{x}(kT) + \left(\int_{kT}^{kT+T} e^{\boldsymbol{F}\eta}d\eta\right)\boldsymbol{G}u(kT)$$

Let's rename $\boldsymbol{\Phi} = e^{\boldsymbol{F}T}$ and $\boldsymbol{\Gamma} = \left(\int_{kT}^{kT+T} e^{\boldsymbol{F}\eta}d\eta\right)\boldsymbol{G}$

Discrete state matrices

So,

$$\boldsymbol{x}(k+1) = \boldsymbol{\Phi}\boldsymbol{x}(k) + \boldsymbol{\Gamma}u(k)$$
$$y(k) = \mathbf{H}\boldsymbol{x}(k) + \mathbf{J}\boldsymbol{u}(k)$$

Again, $\boldsymbol{x}(k+1)$ is shorthand for $\boldsymbol{x}(kT+T)$

Note that we can also write $\boldsymbol{\Phi}$ as:

$$\boldsymbol{\Phi} = \mathbf{I} + \mathbf{F}T\boldsymbol{\Psi}$$

where

$$\boldsymbol{\Psi} = \mathbf{I} + \frac{\mathbf{F}T}{2!} + \frac{\mathbf{F}^2T^2}{3!} + \cdots$$

Simplifying calculation

- We can also use $\boldsymbol{\Psi}$ to calculate $\boldsymbol{\Gamma}$
  - Note that:

$$\Gamma = \sum_{k=0}^{\infty} \frac{\mathbf{F}^kT^k}{(k+1)!}T\mathbf{G}$$
$$= \boldsymbol{\Psi}T\mathbf{G}$$

$\boldsymbol{\Psi}$ itself can be evaluated with the series:

$$\boldsymbol{\Psi} \cong \mathbf{I} + \frac{\mathbf{F}T}{2}\left\{\mathbf{I} + \frac{\mathbf{F}T}{3}\left[\mathbf{I} + \cdots \frac{\mathbf{F}T}{n-1}\left(\mathbf{I} + \frac{\mathbf{F}T}{n}\right)\right]\right\}$$

State-space z-transform

We can apply the z-transform to our system:

$$(z\mathbf{I} - \mathbf{\Phi})\boldsymbol{X}(z) = \mathbf{\Gamma} U(k)$$
$$Y(z) = \mathbf{H}\boldsymbol{X}(z)$$

which yields the transfer function:

$$\frac{Y(z)}{\boldsymbol{X}(z)} = G(z) = \mathbf{H}(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma}$$

State-space control design

- Design for discrete state-space systems is just like the continuous case.
  - Apply linear state-variable feedback:

$$u = -\mathbf{K}\boldsymbol{x}$$

such that $\det(z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma K}) = \alpha_c(z)$

where $\alpha_c(z)$ is the desired control characteristic equation

Predictably, this requires the system controllability matrix $\boldsymbol{\mathcal{C}} = [\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \quad \mathbf{\Phi}^2\mathbf{\Gamma} \quad \cdots \quad \mathbf{\Phi}^{n-1}\mathbf{\Gamma}]$ to be full-rank.

Announcements:

- Final Exam:
  ➔Saturday, June 15 at 9:30 AM (sorry!)

ELEC3004	Signals, Systems & Control	15/06/2013	9:30 AM		Agarwal, Atu - Vasuian, Fab	UQ Union Complex (21) - Innes Room
					Vermeulen, Nat - Zulkarnain, Moh	UQ Union Complex (21) - Heath Room

- Problem Set 2 is due this Friday!

Next Time in Linear Systems ….

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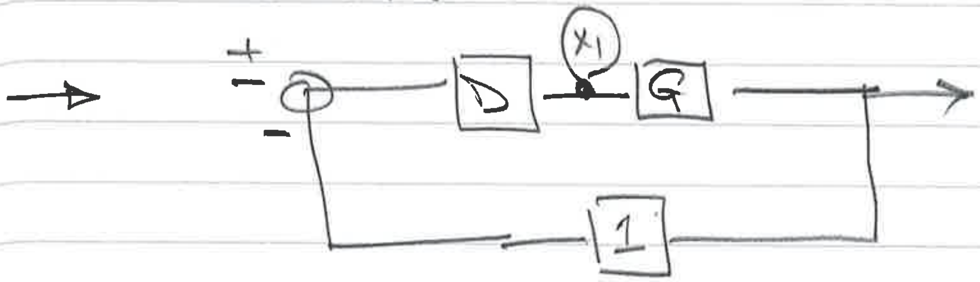
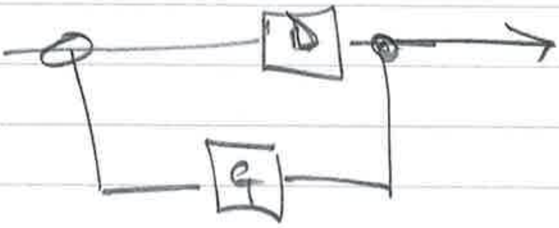
- AKA: I can see that. Yes, I can control that!

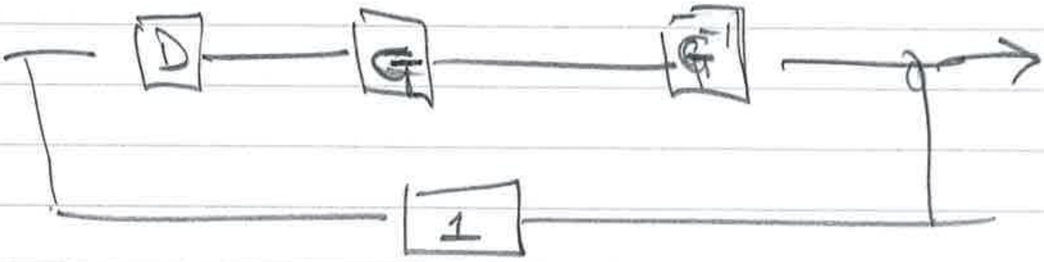
→ SPNS - Lecture 23 — 5/24/2013 —

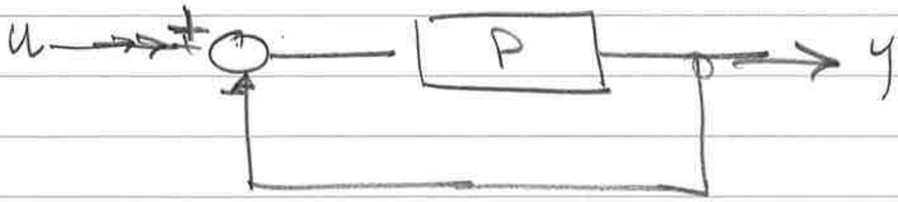
→ internal states:





or inverse plant





$$\rightarrow \boxed{y = \frac{P}{1+P}}$$

$$u_1 = u - y_1$$

$$y = y_1$$

$$P: \quad \dot{x}_1 = A_1 x_1 + B_1 u_1$$

$$y_1 = C_1 x_1 + D_1 u_1$$

$$\rightarrow \dot{x}_1 = \left(A_1 - B_1 (I+D_1)^{-1} C_1\right) x_1 + B_1 \left(I - (I+D_1)^{-1} D_1\right) u$$

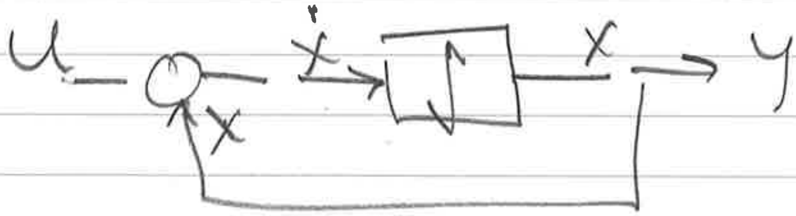
$$y = (I+D_1)^{-1} C_1 x_1 + (I+D_1)^{-1} D_1 u$$

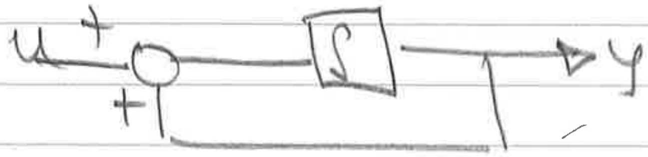
→ STATE-SPACE DECOMPOSITION

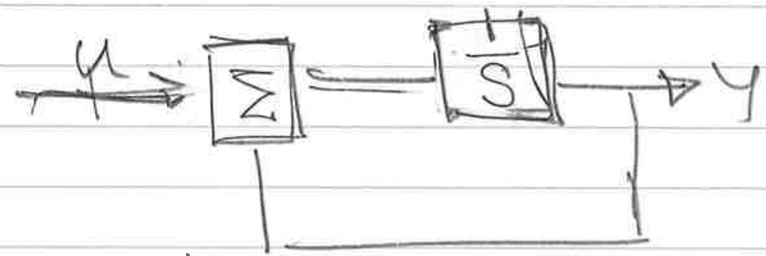
(1) $\dot{x} = x + u$

(2) $y = x$

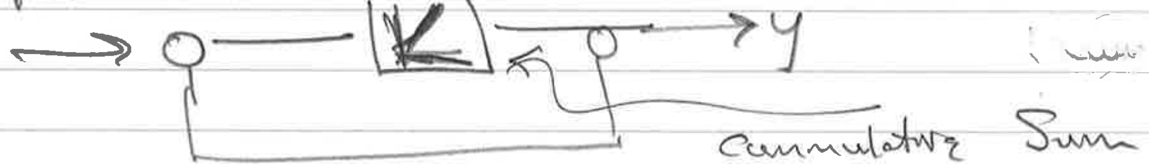
$y = \int [\dot{x}]$



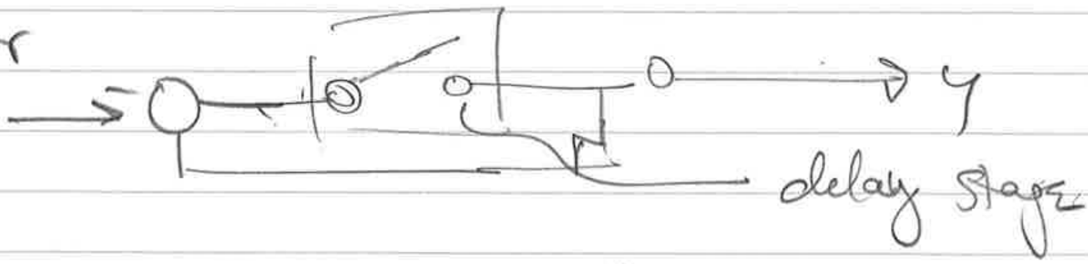




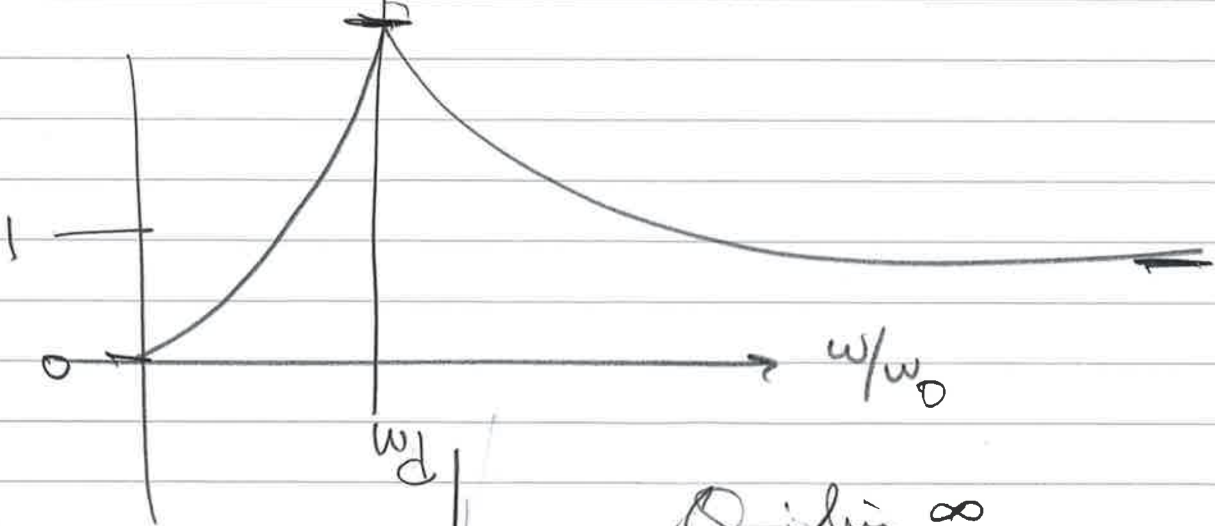
compare to:



or

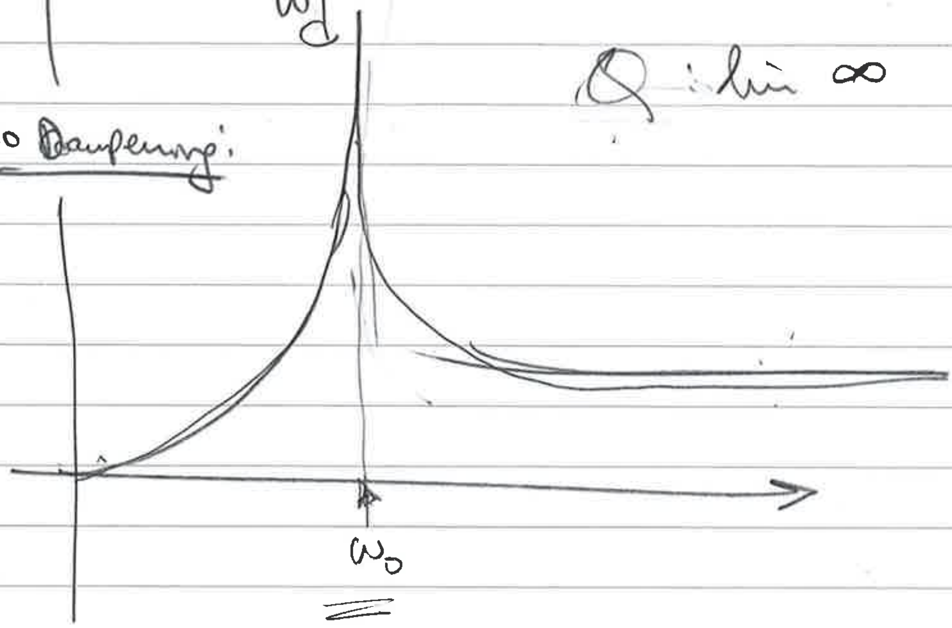


Q-FACTOR for a Resonator



Q : lim ∞

No Dampening:



$\frac{1}{2} \cdot \dot{X}^T m \dot{X} \rightarrow \frac{1}{2} M\dot{X}^2$