Introduction to Digital Control & Stability of Digital Systems

ELEC 3004: **Systems**: Signals & Controls
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(Some material adapted from Paul Pounds)

Lecture 19

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May 10, 2013

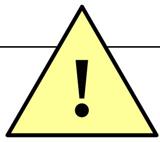
Today…

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Signals & Signal Models
	8-Mar	System Models
3	13-Mar	Linear Dynamical Systems
	15-Mar	Sampling & Data Acquisition
4	20-Mar	Time Domain Analysis of Continuous Time Systems
	22-Mar	System Behaviour & Stability
5	27-Mar	Signal Representation
	29-Mar	Holiday
6	10-Apr	Frequency Response
	12-Apr	z-Transform
7	17-Apr	Noise & Filtering
	19-Apr	Analog Filters
8	24-Apr	Discrete-Time Signals
	26-Apr	Discrete-Time Systems
9	1-May	Digital Filters & IIR/FIR Systems
	3-May	Fourier Transform & DTFT
10	8-May	Introduction to Digital Control
	10-May	Stability of Digital Systems
11	15-May	PID & Computer Control
	17-May	Applications in Industry
12	22-May	State-Space
	24-May	Controllability & Observability
13	29-May	Information Theory/Communications & Review
	31-May	Summary and Course Review

Goals for the Week

- Wrap up Digital Filters (and bookend it with a Pop-Quiz)
- Introduce Digital Systems and Feedback Control
- z-Transform for Digital Control
- Design Using Digital Equivalents ➔ Friday
- Stability of Digital Systems

Announcements:



- Final Exam:
  - 15 Questions (60% Short Answer, 40% Regular Problems)
  - 3 Hours
  - Closed-book
  - Took tutor ~90min to complete
  - Equation sheet will be provided (in addition to your own) [See Prac Final – Coming out next week]
  - Yes, it has an unexpected twist at the end, but you'll like it. ☺

➔Saturday, June 15 at 9:30 AM (sorry!)

ELEC3004	Signals, Systems & Control	15/06/2013	9:30 AM		Agarwal, Atu - Vasuian, Fab	UQ Union Complex (21) - Innes Room
					Vermeulen, Nat - Zulkarnain, Moh	UQ Union Complex (21) - Heath Room

Digital control

Once upon a time…

- Electromechanical systems were controlled by electromechanical compensators
  - Mechanical flywheel governors, capacitors, inductors, resistors, relays, valves, solenoids (fun!)
  - But also complex and sensitive!

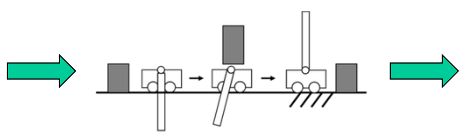
→Idea: Digital computers in real-time control

- Transform approach (classical control)
  - Root-locus methods (pretty much the same as METR 3200)
  - Bode's frequency response methods (these change compared to METR 3200)
- State-space approach (modern control)

→ Model Making: Control of frequency response as well as Least Squares Parameter Estimation

Digital Control

Wikipedia, Cart and pole



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position:

$$v_1^2 = \dot{x}^2$$
$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

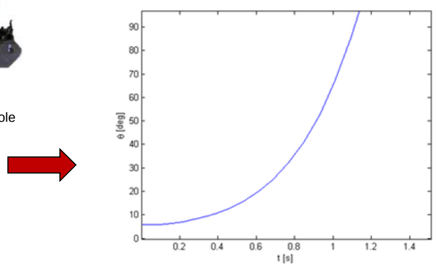
$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

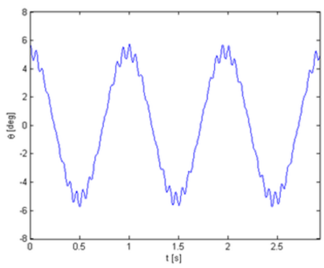
and the equations of motion are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

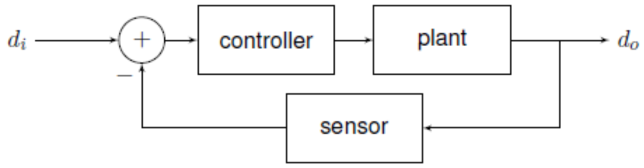
substituting L in these equations and simplifying leads to the equations that describe the motion o

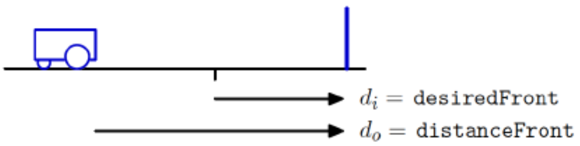
$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$
$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$





Simple Controller Goes Digital





plant: $y[n] = y[n-1] - Tu[n-1]$

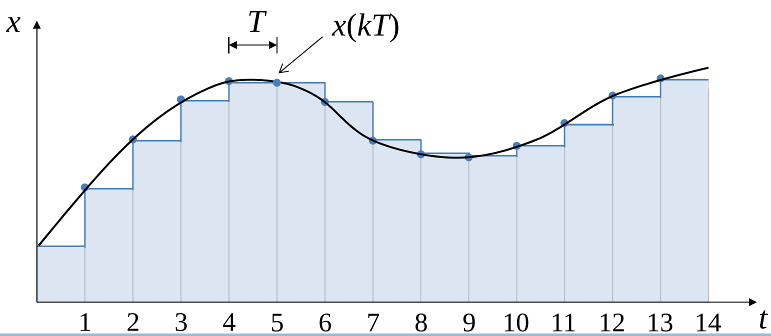
sensor: $y[n] = u[n-1]$

controller: $y[n] = Ku[n]$

Complex system behaviors, depending on K

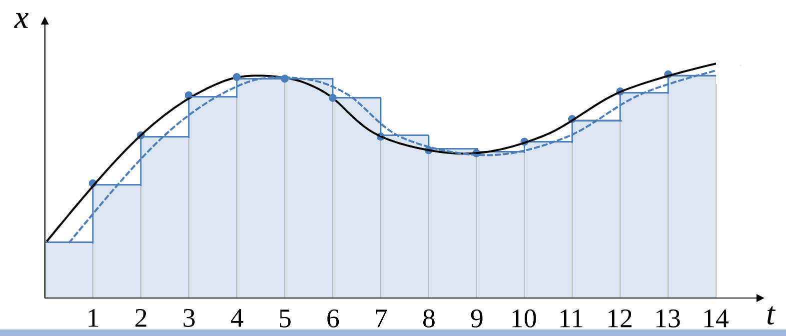
Return to the discrete domain

- Recall that continuous signals can be represented by a series of samples with period T



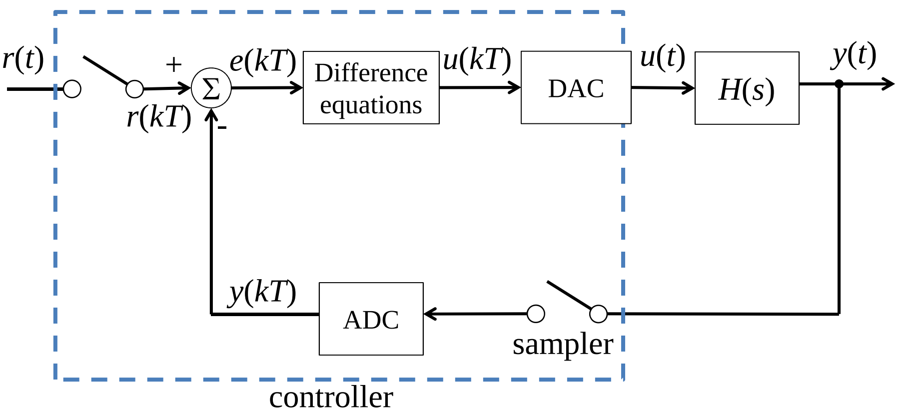
Zero Order Hold

- An output value of a synthesised signal is held constant until the next value is ready
  - This introduces an effective delay of $T/2$



Digitisation

- Continuous signals sampled with period T
- kth control value computed at $t_k = kT$



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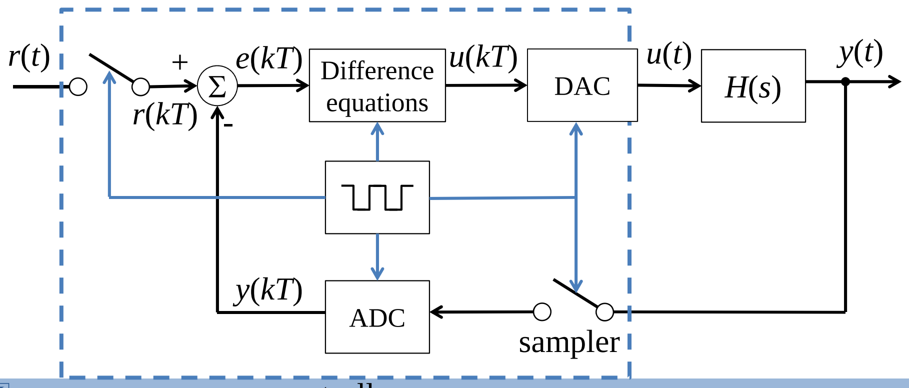


Difference equations

- How to represent differential equations in a computer? Difference equations!
- The output of a difference equation system is a function of current and previous values of the input and output:

$$y(t_k) = D(x(t_k), x(t_{k-1}), \dots, x(t_{k-n}), y(t_{k-1}), \dots, y(t_{k-n}))$$

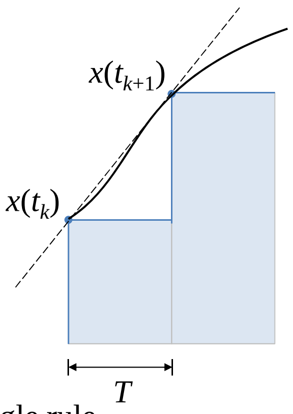
  – We can think of x and y as parameterised in k

    Useful shorthand: $x(t_{k+i}) \equiv x(k+i)$

Euler's method*

- Dynamic systems can be approximated[†] by recognising that:

$$\dot{x} \cong \frac{x(k+1) - x(k)}{T}$$



- As $T \to 0$, approximation error approaches 0

*Also known as the forward rectangle rule
†Just an approximation – more on this later

An example!

Convert the system $\frac{Y(s)}{X(s)} = \frac{s+2}{s+1}$ into a difference equation with period T, using Euler's method.

1. Rewrite the function as a dynamic system:

$$sY(s) + Y(s) = sX(s) + 2X(s)$$

Apply inverse Laplace transform:

$$\dot{y}(t) + y(t) = \dot{x}(t) + 2x(t)$$

2. Replace continuous signals with their sampled domain equivalents, and differentials with the approximating function

$$\frac{y(k+1) - y(k)}{T} + y(k) = \frac{x(k+1) - x(k)}{T} + 2x(k)$$

An example!

Simplify:

$$y(k+1) - y(k) + Ty(k) = x(k+1) - x(k) + 2Tx(k)$$
$$y(k+1) + (T-1)y(k) = x(k+1) + (2T-1)x(k)$$

$$y(k+1) = x(k+1) + (2T-1)x(k) - (T-1)y(k)$$

We can implement this in a computer.

Cool, let's try it!

Back to the future

A quick note on causality:

- Calculating the "(k+1)th" value of a signal using

$$y(k+1) = x(k+1) + Ax(k) - By(k)$$

relies on also knowing the next (future) value of $x(t)$.

(this requires very advanced technology!)

future value current values

- Real systems always run with a delay:

$$y(k) = x(k) + Ax(k-1) - By(k-1)$$

Back to the example!

```
T = 0.02;  //period of 50 Hz, a number pulled from thin air
A = 2*T-1; //pre-calculated control constants
B = T-1;

...

while(1)
   {
      if(interrupt_flag)          //this triggers every 20 ms
      {
         x0 = x;                  //save previous values
         y0 = y;
         x = update_input();      //get latest x value
         y = x + A*x0 - B*y0;     //do the difference equations
         update_output(y);        //write out current value
      }
   }
```

(The actual maths bit)

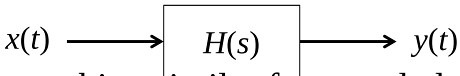
Coping with complexity

- Transfer functions help control complexity
  - Recall the Laplace transform:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s)$$

where

$$\mathcal{L}\{\dot{f}(t)\} = sF(s)$$



Is there a something similar for sampled systems?

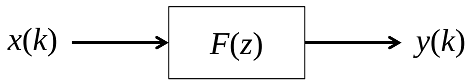
The *z*-transform

- The discrete equivalent is the *z*-Transform†:

$$\mathcal{Z}\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$$

and

$$\mathcal{Z}\{f(k-1)\} = z^{-1}F(z)$$



Convenient!

†This is not an approximation, but approximations are easier to derive

The *z*-transform

- Some useful properties
  - **Delay by *n* samples**: $\mathcal{Z}\{f(k-n)\} = z^{-n}F(z)$
  - **Linear**: $\mathcal{Z}\{af(k) + bg(k)\} = aF(z) + bG(z)$
  - **Convolution**: $\mathcal{Z}\{f(k) * g(k)\} = F(z)G(z)$

So, all those block diagram manipulation tools you know and love will work just the same!

The *z*-transform

- In practice, you'll use look-up tables or computer tools (ie. Matlab) to find the *z*-transform of your functions

$F(s)$	$F(kt)$	$F(z)$
$\frac{1}{s}$	1	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\frac{1}{s^2+a^2}$	$\sin(akT)$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$

Why z-Transform

- Makes it easy to analyse feedback systems governed by difference equations
- For any complex number $z = re^{j\omega}$ $y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$
- Forward Analysis: $Y(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$
- Backward Synthesis
  (for any fixed r > 0 on which the Z-transform converges):

$$y[n] = \frac{1}{2\pi} \int_{2\pi} Y(re^{j\omega})(re^{j\omega})^n d\omega$$

z-Transforms for Difference Equations

- First-order linear constant coefficient difference equation:

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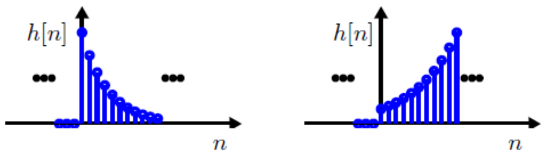
$$y[n] = ay[n-1] + bu[n]$$



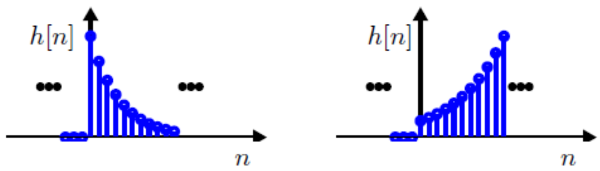
$$h[n] = \begin{cases} ba^n & n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H(z) = \sum_{k=0}^{\infty} ba^k z^{-k} = b \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \frac{b}{1 - az^{-1}}, \quad \text{when } |z| > |a|.$$

z-Transforms for Difference Equations

First-order linear constant coefficient difference equation:

$$y[n] = ay[n-1] + bu[n]$$



$$y[n] - ay[n-1] = bu[n]$$

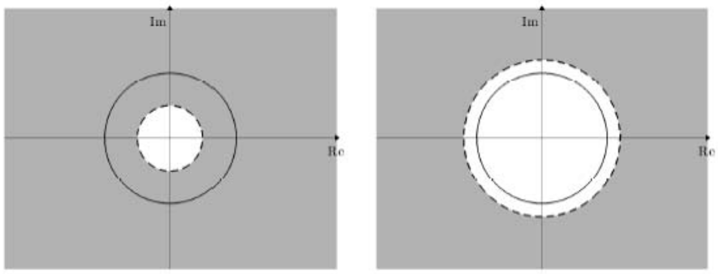
$$\updownarrow$$

$$Y(z) - az^{-1}Y(z) = bU(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b}{1 - az^{-1}}, \text{ when does it converge?}$$

Region of Convergence (ROC) Plots

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b}{1 - az^{-1}}, \quad |z| > |a|$$



$a = .5$ $\qquad$ $a = 1.2$

Properties of the ROC

➔The ROC is always defined by circles centered around the origin.

$h[k]r^{-k}$ is absolutely summable, where $r = |z|$.

➔Right-sided signals have "outsided" ROCs.

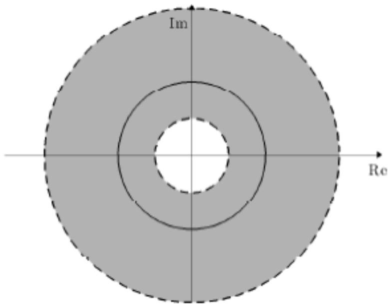
if $\exists n_0$ such that $h[n] = 0\ \forall n < n_0$, then if $r_0 \in$ ROC, then $\forall r$ with $r_0 < r < \infty$ are also in the ROC.

➔Left-sided signals have "insided" ROCs. (with $\forall$r within $0<r<r_0$)

Combinations of Signals

$$y_1[n] = \begin{cases} ba^n & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad y_2[n] = \begin{cases} 0 & n \geq 0 \\ -ba^n & n < 0 \end{cases}$$

$a = .5$ $\qquad$ $a = 2$



ROC for $\alpha_1 y_1[n] + \alpha_2 y_2[n]$

Higher-order difference equations

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + a_3 y[n-3] + b_0 u[n] + b_1 u[n-1] + \ldots$$

Easy to take the Z-transform

$$Y(z) = a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + a_3 z^{-3} Y(z) + b_0 U(z) + \ldots$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-1} + \ldots}{1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3} + \ldots}$$

Final value theorem

- An important question: what is the steady-state output a stable system at $t = \infty$?
  - For continuous systems, this is found by:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

  - The discrete equivalent is:

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1} (1 - z^{-1})X(z)$$

(Provided the system is stable)

An example!

- Back to our difference equation:

$$y(k) = x(k) + Ax(k-1) - By(k-1)$$

becomes

$$Y(z) = X(z) + Az^{-1}X(z) - Bz^{-1}Y(z)$$
$$(z + B)Y(z) = (z + A)X(z)$$

which yields the transfer function:

$$\frac{Y(z)}{X(z)} = \frac{z + A}{z + B}$$

Note: It is also not uncommon to see systems expressed as polynomials in z^{-n}

This looks familiar…

- Compare:

$$\frac{Y(s)}{X(s)} = \frac{s+2}{s+1} \text{ vs } \frac{Y(z)}{X(z)} = \frac{z+A}{z+B}$$

How are the Laplace and *z* domain representations related?

Consider the simplest system

- Take a first-order response:

$$f(t) = e^{-at} \Rightarrow \mathcal{L}\{f(t)\} = \frac{1}{s+a}$$

- The discrete version is:

$$f(kT) = e^{-akT} \Rightarrow \mathcal{Z}\{f(k)\} = \frac{z}{z - e^{-aT}}$$

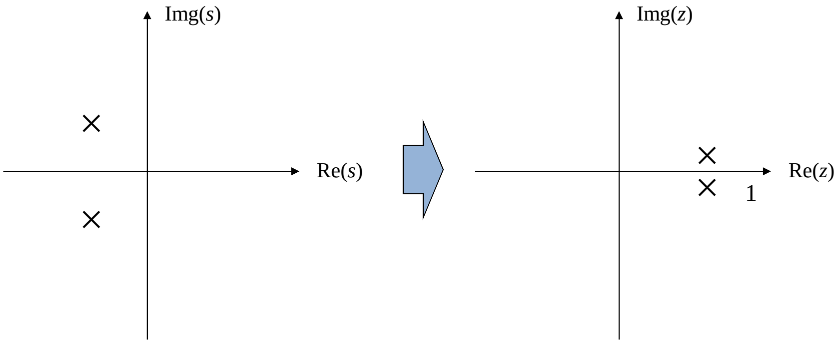
The equivalent system poles are related by

$$z = e^{sT}$$

That sounds somewhat profound… but what does it mean?

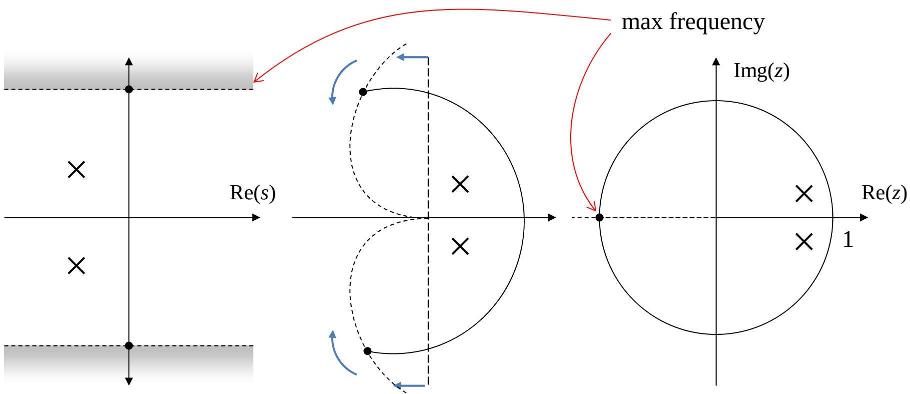
The *z*-Plane

- *z*-domain poles and zeros can be plotted just like *s*-domain poles and zeros:



Deep insight #1

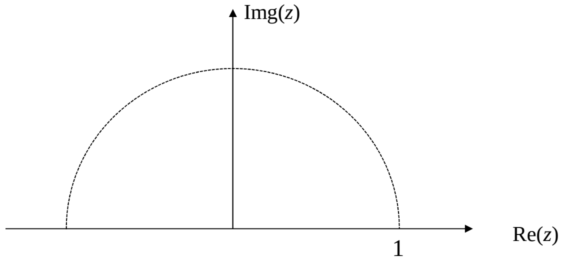
The mapping between continuous and discrete poles and zeros acts like a distortion of the plane



The *z*-plane

- We can understand system response by pole location in the *z*-plane

[Adapted from Franklin, Powell and Emami-Naeini]



Effect of pole positions

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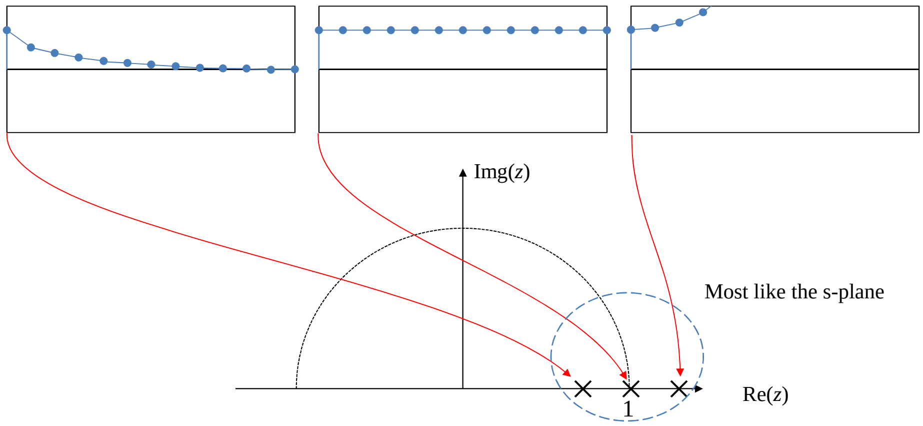

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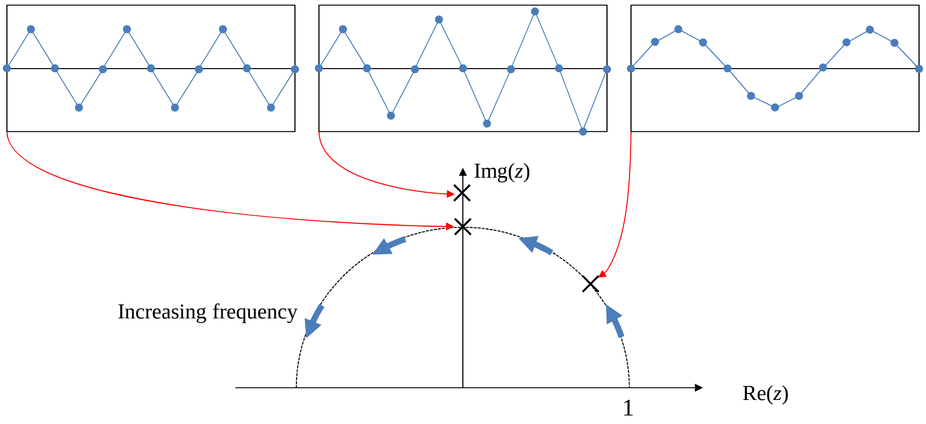

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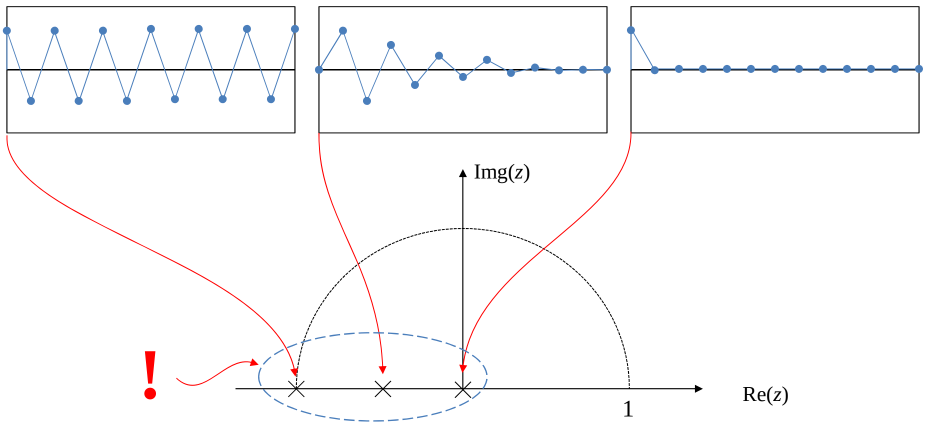
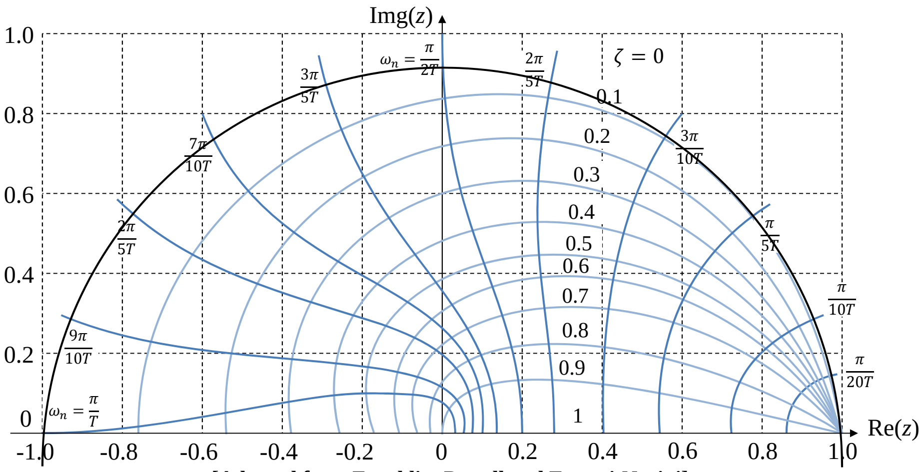


Damping and natural frequency

$$z = e^{sT} \text{ where } s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

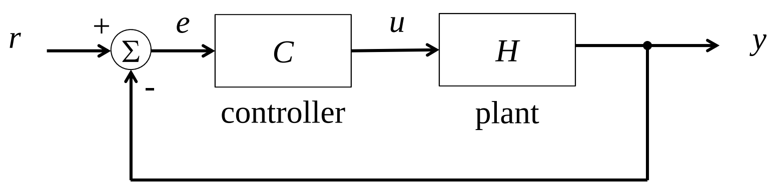


[Adapted from Franklin, Powell and Emami-Naeini]

Quick refresher: the root locus

- The transfer function for a closed-loop system can be easily calculated:

$$y = CH(r-y)$$
$$y + CHy = CHr$$
$$\therefore \frac{y}{r} = \frac{CH}{1+CH}$$



Quick refresher: the root locus

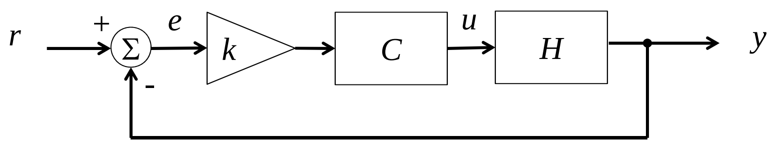
- We often care about the effect of increasing gain of a control compensator design:

$$\frac{y}{r} = \frac{kCH}{1 + kCH}$$

Multiplying by denominator:

$$\frac{y}{r} = \frac{kC_nH_n}{C_dH_d + kCnHn}$$

characteristic polynomial



Example:

- Is this system stable?

$$u(k) = 0.9u(k-1) - 0.2u(k-2)$$

- Time-shift it:

$$u(k+2) = 0.9u(k+1) - 0.2u(k)$$

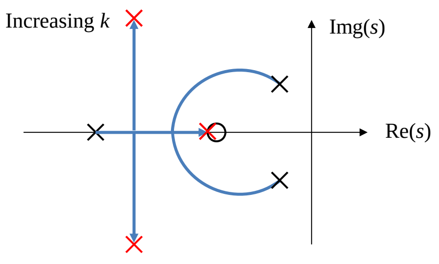
- z-Transform:

$$(1)z^2 - 0.9z + 0.2 = 0$$

- Characteristic Roots:
  z=0.5, z=0.4 ➔ STABLE!

Quick refresher: the root locus

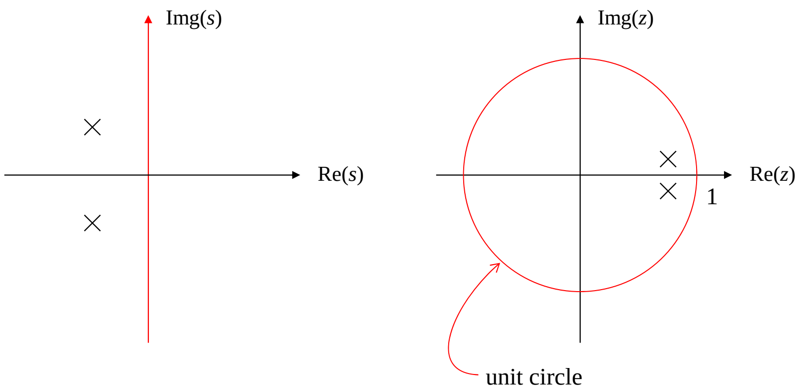
- Pole positions change with increasing gain
  - The trajectory of poles on the pole-zero plot with changing k is called the "root locus"
  - This is sometimes quite complex



(In practice you'd plot these with computers)

z-plane stability

- In the z-domain, the unit circle is the system stability bound



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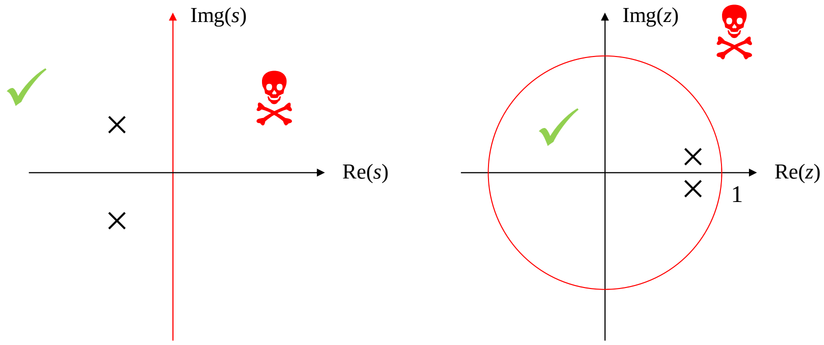
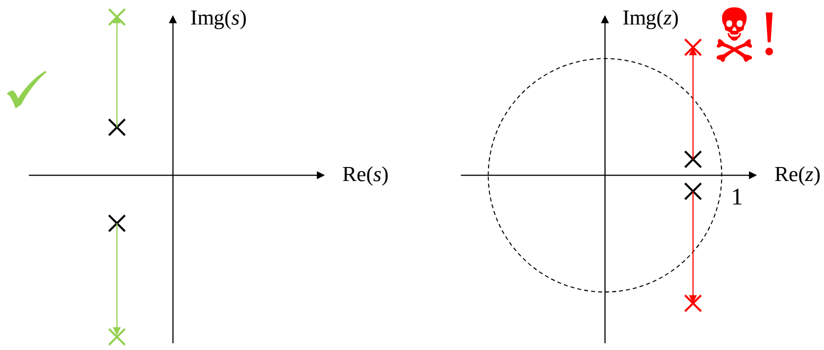


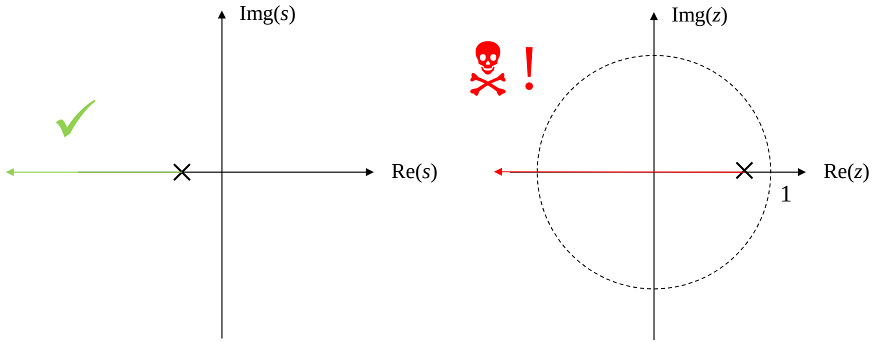
z-plane stability

- The z-plane root-locus in closed loop feedback behaves just like the s-plane:



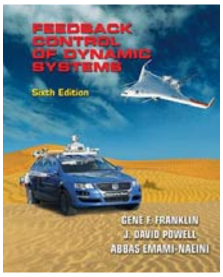
Deep insight #2

Gains that stabilise continuous systems can actually *destabilise* digital systems!



Quick plug*

- Most of this is based on Chapter 8 of "**Feedback Control of Dynamic Systems**" by Franklin, Powell and Emami-Naeini.



* No, they're not paying me – it's just a really good book!