Discrete Time Systems

ELEC 3004: **Systems**: Signals & Controls

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(Some material adapted from courses by Russ Tedrake, MIT)

Lecture 15

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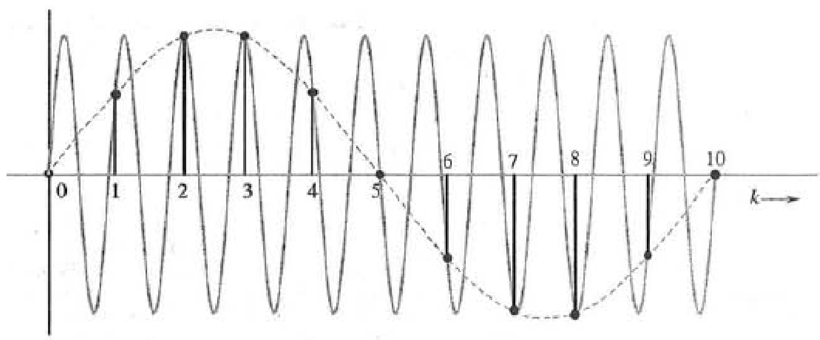
Goals for the Week

- Properties of Discrete Time [DT] Signals
- DT Signal Models
- DT Signal Operations
- DT Convolution
- DT Systems ➔ Friday
- (also for Lab 3 / Exp 4): Introduce FIR Filters ➔ Friday

Today…

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Signals & Signal Models
	8-Mar	System Models
3	13-Mar	Linear Dynamical Systems
	15-Mar	Sampling & Data Acquisition
4	20-Mar	Time Domain Analysis of Continuous Time Systems
	22-Mar	System Behaviour & Stability
5	27-Mar	Signal Representation
	29-Mar	Holiday
6	10-Apr	Frequency Response
	12-Apr	z-Transform
7	17-Apr	Noise & Filtering
	19-Apr	Analog Filters
8	24-Apr	Discrete-Time Signals
	26-Apr	Discrete-Time Systems
9	1-May	Digital Filters & IIR/FIR Systems
	3-May	Fourier Transform & DTFT
10	8-May	State-Space
	10-May	Controllability & Observability
11	15-May	Introduction to Digital Control
	17-May	Stability of Digital Systems
12	22-May	PID & Computer Control
	24-May	Information Theory & Communications
13	29-May	Applications in Industry
	31-May	Summary and Course Review

Recap: Aliasing
Another view of this



Recap: Alliasing

- Aliasing - through sampling, two entirely different analog sinusoids take on the same "discrete time" identity

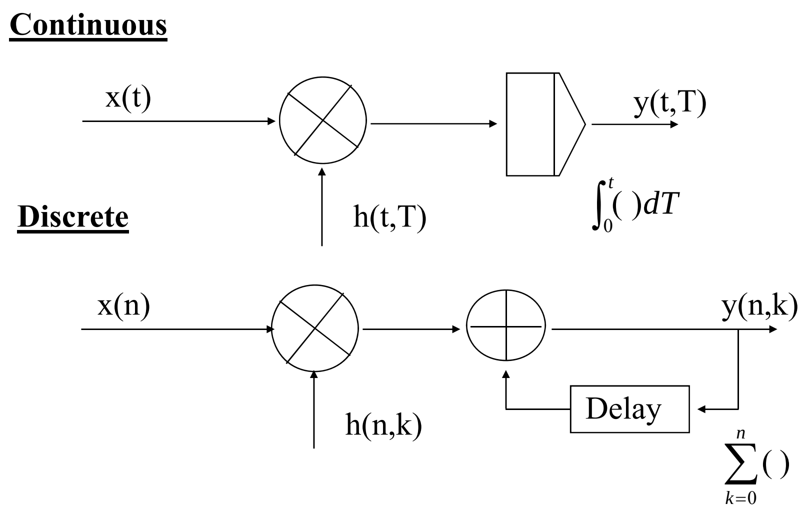
For f[k]=cosΩk, Ω=ωT:

The period has to be less than Fh (highest frequency): $T \leq \frac{1}{2\mathcal{F}_h}$

Thus: $0 \leq \mathcal{F} \leq \frac{\mathcal{F}_s}{2}$

ω_f: aliased frequency: $\omega T = \omega_f T + 2\pi m$

Basic Signal Processing: Continuous & Discrete



Discrete LTI

A system G is time-invariant if

$$y[k] = Gu[k] \quad \Rightarrow \quad y[k-k_0] = Gu[k-k_0]$$

Interpretations:

- "shifting the input shifts the output"
- "absolute time/location does not matter"
- "the system behaves the same at all times"

Discrete LTI ➔ Circulant Systems

- Circulant matrices are the "finite" equivalent of LTI systems.
- Their properties are very similar to those of LTI systems.
- The mathematics are a bit simpler, just "standard" linear algebra.
- In particular, they have a nice and simple factorization structure.

A circulant matrix is fully defined by the entries of the first row (or column):

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$

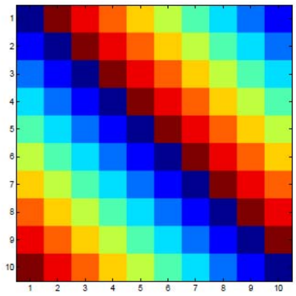
We use the notation $C(a_0, a_1, \ldots, a_{n-1})$ or simply $C(a)$ to denote this matrix.

For instance, $C(1, 0, 0, \ldots, 0) = I_n$.

Circulant Systems

A square $n \times n$ matrix A is *circulant* if it has the form

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$



- In general, matrices do not commute (AB ≠ BA)
- Any two circulant matrices do commute!

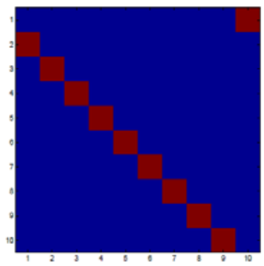
$$C(a)C(b) = C(b)C(a)$$

Shift matrix Z

- A special circulant matrix

The shift matrix $Z = C(0, 0, 0, \ldots, 0, 1)$.

$$Z = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$



Multiplying by Z cyclically shifts the input:

$$Z \cdot [u_0, u_1, u_2, u_3, \ldots, u_{n-1}]^T = [u_{n-1}, u_0, u_1, u_2, \ldots, u_{n-2}]^T.$$

Shift invariant systems

- The shift Z acts on signals like a unit delay (but it "wraps around")

$$Z \cdot [u_0, u_1, u_2, u_3, \ldots, u_{n-1}]^T = [u_{n-1}, u_0, u_1, u_2, \ldots, u_{n-2}]^T.$$

- Recall time-invariance: "shifting the input shifts the output"
- A natural notion: linear shift-invariant systems

$$D \cdot (Zu) = Z \cdot (Du)$$

or equivalently, DZ = ZD ("system commutes with the shift")

Shift invariant systems are circulants

A linear system D is shift-invariant if $DZ = ZD$.

What do such D look like?

Exercise: $DZ = ZD$ if and only if D is circulant.

Thus, linear shift-invariant systems are given by circulant matrices.

They are the "finite" equivalent of LTI systems.

Examples: blurring system, etc.

Eigenvalues of DT-LTI Systems

- Eigenvalues (λ) and Characteristic Functions:

$$F\left\{x\left(\bullet\right)\right\} = \lambda x\left(\bullet\right)$$

- For DT LTI the eigenfunctions are complex exponentials

$$y\left[n\right] = F\left\{z^n\right\} = \sum_{k=-\infty}^{\infty} h\left[k\right] z^{n-k}$$

$$= \left[\sum_{k=-\infty}^{\infty} h\left[k\right] z^{-k}\right] z^n = H\left(z\right) z^n = \lambda z^n$$

Back to Circulant Matrices… Their **eigenvectors** do not depend on a_i

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$

- Every circulant matrix has an eigenvalue decomposition:

$$C(a) = T \cdot D(a) \cdot T^{-1}$$

$$T_{j,k} = \omega^{j \cdot k} \text{ (for } j,k = 0,1,\ldots,n-1)$$

➔ eigenvectors: $v_k = [1; \omega^k; \omega^{2k}; \ldots; \omega^{(n-1)k}]$

- Notice that the **eigenvectors** **T** do not depend on entries a_i!
- This change-of-base matrix **T** is called the Fourier matrix.

Eigenvalues of Circulant Matrices

- They **do** depend on the values of a_i (a_0, a_1, …, a_{n-1}). (you'd hope)

The eigenvalues are

$$\lambda_k = a_0 + a_1\omega^k + a_2\omega^{2k} + \cdots + a_{n-1}\omega^{(n-1)k},$$

Ex:

$$C(2,5,6,1) = \begin{bmatrix} 2 & 5 & 6 & 1 \\ 1 & 2 & 5 & 6 \\ 6 & 1 & 2 & 5 \\ 5 & 6 & 1 & 2 \end{bmatrix}$$

Then ω is a fourth root of unity, i.e., $\omega = e^{2\pi i/4} = e^{i\pi/2} = i$.

The eigenvalues and eigenvectors are then (verify!):

$$T = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}, \qquad D = \begin{bmatrix} 14 & 0 & 0 & 0 \\ 0 & -4+4i & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -4-4i \end{bmatrix}$$

WHY?:
These Systems Show Up Everywhere

- Ex: Kinematic (constantly accelerating body) systems:

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Discrete Time Convolution in Matrix Form via the <u>Toeplitz Matrix</u>**

DT Convolution: Vector Sums

- Remember: **<u>Flip → Shift → Slide → Sum</u>**

e.g. convolution

x(n) = 1 2 3 4 5
h(n) = 3 2 1

x(k)	0 0 1 | 2 3 4 5	0 0 1 2 | 3 4 5	0 0 1 2 3 | 4 5
h(n,k)	1 2 3 | 0 0 0 0	0 1 2 3 | 0 0 0	0 0 1 2 3 | 0 0 h(n-k)
y(n,k)	3	2 6	1 4 9
y(n)	3	8	14

Sum over all k

Notice the gain

DT Convolution: Matrix Formulation of Convolution

y H x

$$\begin{bmatrix} 3 \\ 8 \\ 14 \\ 20 \\ 26 \\ 14 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

Toeplitz Matrix

Response of a Discrete-Time LTI System & Convolution Sum

- Impulse Response:

Imaging a discrete-time, LTI System **F** .

It's impulse response is given by: $h[n] = F\{\delta[n]\}$

- Arbitrary Response:

An arbitrary input x[n] can be written: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

(Sampling theorem)

So:

$$y[n] = F\{x[n]\} = F\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]\,[\![F\{\delta[n-k]\}]\!]$$

$$h[n-k] = F\{\delta[n-k]\}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$\therefore$ a DT LTI is completely characterized by its impulse response

Convolution Response

- As with the continuous domain, commutatively, associativity, and distributivity hold…
- Commutativity gives a nice result:

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- Makes Step Responses "Easy":

$$s[n] = h[n]\,u[n-k] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{\infty} h[k]$$

$$s[n] - s[n-1] = h[n]$$

DT Causality & BIBO Stability

- Causality:

$$h[n] = 0,\ n < 0$$

$$\rightarrow y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] \quad \text{or} \quad \Rightarrow y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

- Input is Causal if: $x[n] = 0,\ n < 0$

- Then output is Causal:

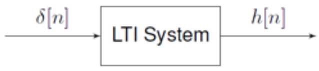
$$y[n] = \sum_{k=0}^{n} h[k]x[n-k] = \sum_{k=0}^{n} x[k]h[n-k]$$

- And, DT LTI is BIBO stable if:

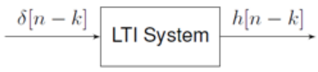
$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Impulse Response (Graphically)

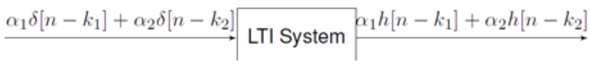
Let's define the *impulse response*, $h[n]$, as the result of applying an LTI system to the unit impulse:

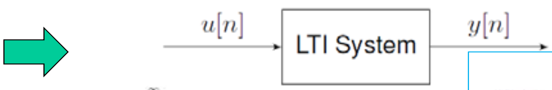


By time invariance, we know



And by linearity, we know





$$u[n] = \sum_{k=-\infty}^{\infty} u[k]\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k]$$

∞ matrix × ∞ vector?

How do you multiply an infinite matrix?

- First let's multiply circulant matrices…
  - A circulant matrix can be descibed completely by its first row or column

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ h & Zh & Z^2h & \cdots & Z^{N-1}h \\ | & | & | & & | \end{bmatrix}$$

Z: Shift operator

- Multiply by u[k] ➔ $\begin{bmatrix} | & | & | & & | \\ h & Zh & Z^2h & \cdots & Z^{N-1}h \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ \vdots \\ u[N-1] \end{bmatrix} = \sum_{k=0}^{N-1} u[k]Z^k h$

∴ For circulant matrices, matrix multiplication reduces to a weighted combination of shifted impulse responses

Two Types of Systems

- Linear shift-invariant:

$$y = \sum_{k=0}^{N-1} u[k] Z^k h$$

Z: Shift operator

$$Z \cdot [u_0, u_1, u_2, u_3, \ldots, u_{n-1}]^T = [u_{n-1}, u_0, u_1, u_2, \ldots, u_{n-2}]^T$$

- Linear time-invariant system

$$y = \sum_{k=-\infty}^{\infty} u[k] R^k h$$

R: Unit delay operator

$$R \cdot [\ldots, u_0, u_1, u_2, u_3, \ldots]^T = [\ldots, u_{-1}, u_0, u_1, \ldots]^T$$

Impulse Response of Both Types

$$y[n] = \frac{1}{2}u[n-1] + \frac{1}{2}u[n]$$

$$\begin{aligned} y[-1] &= 0 \\ y[0] &= \frac{1}{2} \\ y[1] &= \frac{1}{2} \\ y[2] &= 0 \\ &\vdots \end{aligned}$$

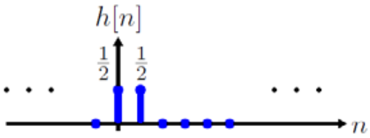
$$y[n] = \frac{1}{2}y[n-1] + u[n]$$

$$\begin{aligned} h[-1] &= 0 \\ h[0] &= 1 \\ h[1] &= \frac{1}{2} \\ h[2] &= \frac{1}{4} \\ &\vdots \end{aligned}$$

$$h[n] = \begin{cases} 0 & n < 0 \\ \left(\frac{1}{2}\right)^n & n \geq 0 \end{cases}$$

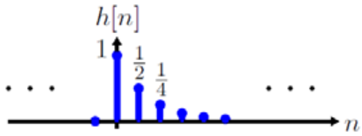
Impulse Response of Both Types

$$y[n] = \frac{1}{2}u[n-1] + \frac{1}{2}u[n]$$



'Finite impulse response" (FIR)

$$y[n] = \frac{1}{2}y[n-1] + u[n]$$



"Infinite impulse response" (IIR)

Next Time in Linear Systems ….

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