Analog Filters

ELEC 3004: **Systems**: Signals & Controls
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Lecture 13

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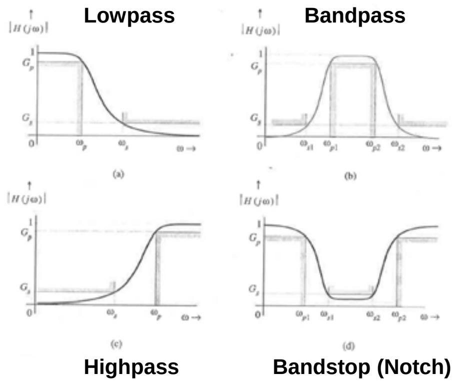
Goals for the Week

- Continue of Discussion of the z-Transform
- Noise
- Introduce Analog Filters (Today) ➔ Friday
  - Frequency Response of an LTIC and LTID System
  - Butterworth
  - Chebyshev
- Digital Filters IIR Filters
    - Design of IIR Filters from Analog Filters
  - FIR Filter

Today…

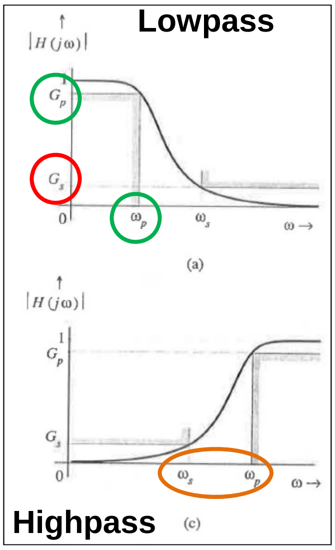
Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Signals & Signal Models
	8-Mar	System Models
3	13-Mar	Linear Dynamical Systems
	15-Mar	Sampling & Data Acquisition
4	20-Mar	Time Domain Analysis of Continuous Time Systems
	22-Mar	System Behaviour & Stability
5	27-Mar	Signal Representation
	29-Mar	Holiday
6	10-Apr	Frequency Response
	12-Apr	z-Transform
7	17-Apr	Noise & Filtering
	19-Apr	Analog Filters
8	24-Apr	Discrete-Time Signals
	26-Apr	Discrete-Time Systems
9	1-May	Digital Filters & IIR/FIR Systems
	3-May	Fourier Transform & DTFT
10	8-May	State-Space
	10-May	Controllability & Observability
11	15-May	Introduction to Digital Control
	17-May	Stability of Digital Systems
12	22-May	PID & Computer Control
	24-May	Information Theory & Communications
13	29-May	Applications in Industry
	31-May	Summary and Course Review

Filters



- *Frequency-shaping filters*: LTI systems that change the shape of the spectrum
- *Frequency-selective filters:* Systems that pass some frequencies undistorted and attenuate others

Filters



Specified Values:

- Gp = minimum passband gain

Typically:

$$G_p = \frac{1}{\sqrt{2}} = -3dB$$

- Gs = maximum stopband gain
  - **Low**, not zero (sorry!)
  - For realizable filters, the gain cannot be zero over a finite band (Paley-Wiener condition)
- Transition Band:

  transition from the passband to the stopband ➔ ωp≠ ωs

Filter Design & z-Transform

Filter Type	Mapping	Design Parameters
Low-pass	$z^{-1} \to \frac{z^{-1}-\alpha}{1-\alpha z^{-1}}$	$\alpha = \frac{\sin[(\omega_c - \omega_c')/2]}{\sin[(\omega_c + \omega_c')/2]}$ ω_c' = desired cutoff frequency
High-pass	$z^{-1} \to -\frac{z^{-1}+\alpha}{1+\alpha z^{-1}}$	$\alpha = -\frac{\cos[(\omega_c + \omega_c')/2]}{\cos[(\omega_c - \omega_c')/2]}$ ω_c' = desired cutoff frequency
Bandpass	$z^{-1} \to -\frac{z^{-2} - [2\alpha\beta/(\beta+1)]z^{-1} + [(\beta-1)/(\beta+1)]}{[(\beta-1)/(\beta+1)]z^{-2} - [2\alpha\beta/(\beta+1)]z^{-1} + 1}$	$\alpha = \frac{\cos[(\omega_{c2} + \omega_{c1})/2]}{\cos[(\omega_{c2} - \omega_{c1})/2]}$ $\beta = \cot[(\omega_{c2} - \omega_{c1})/2]\tan(\omega_c/2)$ ω_{c1} = desired lower cutoff frequency ω_{c2} = desired upper cutoff frequency
Bandstop	$z^{-1} \to \frac{z^{-2} - [2\alpha/(\beta+1)]z^{-1} + [(1-\beta)/(1+\beta)]}{[(1-\beta)/(1+\beta)]z^{-2} - [2\alpha/(\beta+1)]z^{-1} + 1}$	$\alpha = \frac{\cos[(\omega_{c1} + \omega_{c2})/2]}{\cos[(\omega_{c1} - \omega_{c2})/2]}$ $\beta = \tan[(\omega_{c2} - \omega_{c1})/2]\tan(\omega_c/2)$ ω_{c1} = desired lower cutoff frequency ω_{c2} = desired upper cutoff frequency

Butterworth Filters

- Butterworth: Smooth in the pass-band
- The amplitude response $|H(j\omega)|$ of an n[th] order Butterworth low pass filter is given by:

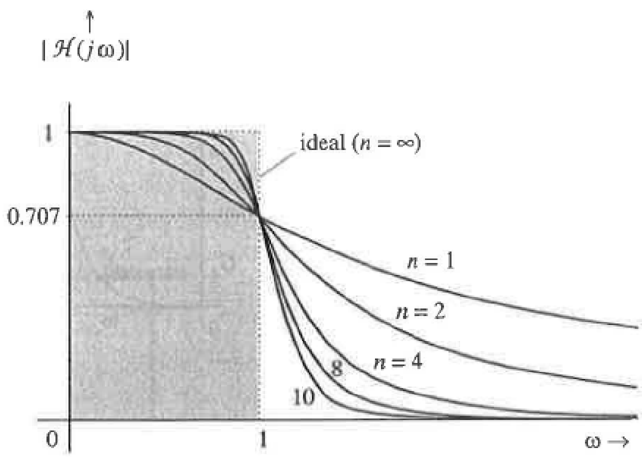
$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

- The normalized case (ω_c=1)

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}} \Rightarrow \mathcal{H}(j\omega)\mathcal{H}(-j\omega) = |\mathcal{H}(j\omega)|^2 = \frac{1}{1+\omega^{2n}}$$

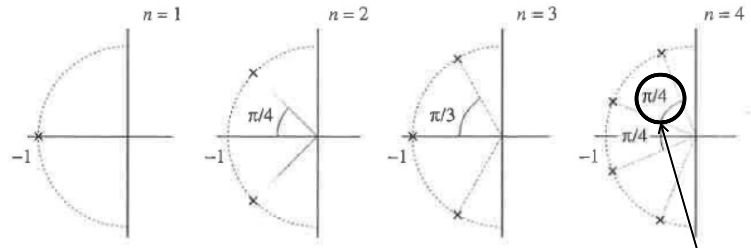
Recall that: $|H(j\omega)|^2 = H(j\omega)\,H(-j\omega)$

Butterworth Filters



Butterworth Filters of Increasing Order: Seeing this Using a Pole-Zero Diagram

- Increasing the order, increases the number of poles:



➔Odd orders (n=1,3,5…):

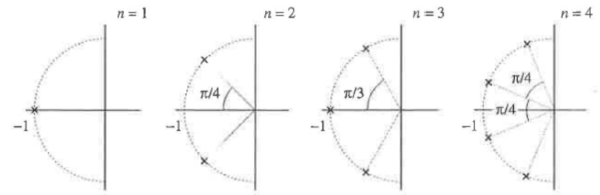
- Have a pole on the Real Axis

➔Even orders (n=2,4,6…):

- Have a pole on the off axis

Angle between poles: $\frac{\pi}{n}$

Butterworth Filters: Pole-Zero Diagram



- Since H(s) is stable and causal, its poles must lie in the LHP
- Poles of -H(s) are those in the RHP
- Poles lie on the unit circle (for a normalized filter)

n is the order of the filter

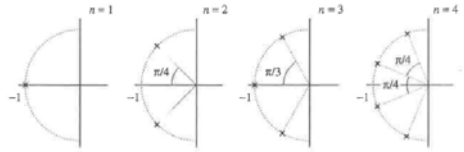
➔$H(s) = \frac{1}{(s-s_1)(s-s_2)\ldots(s-s_n)}$

Where:

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)}$$

$$= \cos\frac{\pi}{2n}(2k+n-1) + j\sin\frac{\pi}{2n}(2k+n-1) \qquad k = 1, 2, 3, \ldots, n$$

Butterworth Filters: 4th Order Filter Example



- Plugging in for n=4, k=1,…4:

$$\mathcal{H}(s) = \frac{1}{(s+0.3827-j0.9239)(s+0.3827+j0.9239)(s+0.9239-j0.3827)(s+0.9239+j0.3827)}$$
$$= \frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)}$$
$$= \frac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1}$$

- We can generalize ➔ Butterworth Table

n	a_1	a_2	a_3	a_4	a_5
2	1.41421356				
3	2.00000000	2.00000000			
4	2.61312593	3.41421356	2.61312593		
5	3.23606798	5.23606798	5.23606798	3.23606798	
6	3.86370331	7.46410162	9.14162017	7.46410162	3.86370331

This is for 3dB bandwidth at ω_c=1

Butterworth Filters: Scaling Back (from Normalized)

- Start with Normalized equation & Table
- Replace ω with $\frac{\omega}{\omega_c}$ in the filter equation

- For example:
  for f_c=100Hz → ω_c=200π rad/sec

  From the Butterworth table: for n=2, $a_1=\sqrt{2}$
  Thus:

$$H(s) = \frac{1}{(\frac{s}{200\pi})^2+\sqrt{2}(\frac{s}{200\pi})+1}$$
$$= \frac{1}{s^2+200\pi\sqrt{2}+40{,}000\pi^2}$$

Butterworth: Determination of Filter Order

- Define $\mathbf{G_x}$ as the gain of a lowpass Butterworth filter at $\omega = \omega_x$
- Then:

$$\hat{G}_x = 20\log_{10}|H(j\omega_x)| = -10\log\left[1+\left(\frac{\omega_x}{\omega_c}\right)^{2n}\right]$$

And thus:

$$\hat{G}_p = -10\log\left[1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}\right]$$

$$\hat{G}_s = -10\log\left[1+\left(\frac{\omega_s}{\omega_c}\right)^{2n}\right]$$

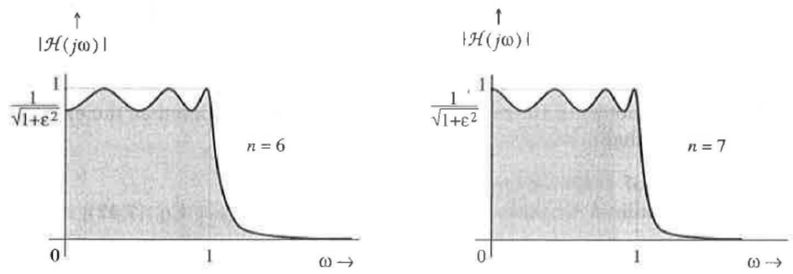
Or alternatively: $\omega_c = \dfrac{\omega_p}{\left[10^{-\hat{G}_p/10}-1\right]^{1/2n}}$ & $\omega_c = \dfrac{\omega_s}{\left[10^{-\hat{G}_s/10}-1\right]^{1/2n}}$

Solving for n gives:

$$n = \frac{\log\left[\left(10^{-\hat{G}_s/10}-1\right)/\left(10^{-\hat{G}_p/10}-1\right)\right]}{2\log(\omega_s/\omega_p)}$$

PS. See Lathi 4.10 (p. 453) for an example in MATLAB

Chebyshev Filters



- **equal-ripple:**
  Because all the ripples in the passband are of equal height
- If we reduce the ripple, the passband behaviour improves, but it does so at the cost of stopband behaviour

Chebyshev Filters

- Chebyshev Filters: Provide tighter transition bands (sharper cutoff) than the same-order Butterworth filter, but this is achieved at the expense of inferior passband behavior (rippling)

➔ For the lowpass (LP) case: at higher frequencies (in the stopband), the Chebyshev filter gain is smaller than the comparable Butterworth filter gain by about ***6(n - 1) dB***

- The amplitude response of a normalized Chebyshev lowpass filter is:

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1+\epsilon^2 {C_n}^2(\omega)}}$$

Where Cn(ω), the nth-order Chebyshev polynomial, is given by:

$$C_n(\omega) = \cos\left(n \cos^{-1} \omega\right)$$

$$C_n(\omega) = \cosh\left(n \cosh^{-1} \omega\right)$$

and where C_n is given by:

n	$C_n(\omega)$
0	1
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$

Normalized Chebyshev Properties

- It's normalized: The passband is 0<ω<1
- **Amplitude response**: has **ripples** in the passband and is **smooth** (monotonic) in the stopband
- **Number of ripples**: there is a total of ***n*** maxima and minima over the passband 0<ω<1
- $C_n^2(0) = \begin{cases} 0, & n: odd \\ 1, & n: even \end{cases}$ ➔ $|H(0)| = \begin{cases} 1, & n: odd \\ \frac{1}{\sqrt{1+\epsilon^2}}, & n: even \end{cases}$
- ϵ: ripple height → $r = \sqrt{1+\epsilon^2}$
- The Amplitude at ω=1: $\frac{1}{r} = \frac{1}{\sqrt{1+\epsilon^2}}$
- For Chebyshev filters, the ripple ***r*** dB takes the place of G_p

Determination of Filter Order

- The gain is given by: $\hat{G} = -10\log\left[1+\epsilon^2 {C_n}^2(\omega)\right]$

Thus, the gain at ω_s is: $\epsilon^2 {C_n}^2(\omega_s) = 10^{-\hat{G}_s/10} - 1$

- Solving:

$$n = \frac{1}{\cosh^{-1}(\omega_s)} \cosh^{-1}\left[\frac{10^{-\hat{G}_s/10} - 1}{10^{\hat{r}/10} - 1}\right]^{1/2}$$

- General Case:

$$n = \frac{1}{\cosh^{-1}(\omega_s/\omega_p)} \cosh^{-1}\left[\frac{10^{-\hat{G}_s/10} - 1}{10^{\hat{r}/10} - 1}\right]^{1/2}$$

Chebyshev Pole Zero Diagram

- Whereas Butterworth poles lie on a semi-circle,
  The poles of an n[th]-order normalized Chebyshev filter lie on a **semiellipse** of the major and minor semiaxes:

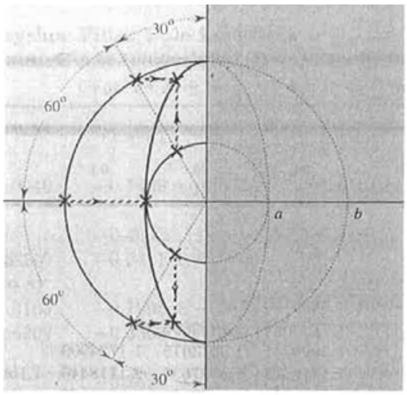
$$a = \sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right) \quad \& \quad b = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right)$$

And the poles are at the locations:

$$H(s) = \frac{1}{(s-s_1)(s-s_2)\dots(s-s_n)}$$

$$s_k = -\sin\left[\frac{(2k-1)\pi}{2n}\right]\sinh x + j\cos\left[\frac{(2k-1)\pi}{2n}\right]\cosh x, \quad k = 1,\dots,n$$

Ex: Chebyshev Pole Zero Diagram for **n=3**



Procedure:

1. Draw two semicircles of radii **a** and **b** (from the previous slide).
2. Draw radial lines along the corresponding Butterworth angles (π/n) and locate the n^{th}-order Butterworth poles (shown by crosses) on the two circles.
3. The location of the k^{th} Chebyshev pole is the intersection of the horizontal projection and the vertical projection from the corresponding k^{th} Butterworth poles on the outer and the inner circle, respectively.

Chebyshev Values / Table

$$\mathcal{H}(s) = \frac{K_n}{C'_n(s)} = \frac{K_n}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$K_n = \begin{cases} a_0 & n \text{ odd} \\ \dfrac{a_0}{\sqrt{1+\epsilon^2}} = \dfrac{a_0}{10^{\hat{r}/20}} & n \text{ even} \end{cases}$$

n	a_0	a_1	a_2	a_3
1	1.9652267			
2	1.1025103	1.0977343		
3	0.4913067	1.2384092	0.9883412	
4	0.2756276	0.7426194	1.4539248	0.9528114

1 db ripple
$(\hat{r} = 1)$

Other Filter Types:
Chebyshev Type II = Inverse Chebyshev Filters

- Chebyshev filters passband has ripples and the stopband is smooth.
- **Instead:** this has **passband** have **smooth** response and **ripples** in the stopband.

➔ Exhibits maximally flat passband response and equi-ripple stopband

➔ `Cheby2` **in MATLAB**

$$|\mathcal{H}(\omega)|^2 = 1 - |\mathcal{H}_C(1/\omega)|^2 = \frac{\epsilon^2 C_n^2(1/\omega)}{1+\epsilon^2 C_n^2(1/\omega)}$$

Where: H_c is the Chebyshev filter system from before

- Passband behavior, especially for small ω, is **better** than Chebyshev
- **Smallest transition band** of the 3 filters (Butter, Cheby, Cheby2)
- Less time-delay (or phase loss) than that of the **Chebyshev**
- Both needs the **same order *n*** to meet a set of specifications.
- **$$$** (or number of elements):
  Cheby < Inverse Chebyshev < Butterworth (of the same **performance** [not order])

Other Filter Types:
Elliptic Filters (or Cauer) Filters

- Allow **ripple** in **both** the passband and the stopband,
  ➔ we can achieve **tighter** transition band

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1+\epsilon^2 R_n{}^2(\omega)}}$$

Where: R_n is the n[th]-order Chebyshev rational function determined from a given ripple spec.
ϵ controls the ripple
$Gp = \frac{1}{\sqrt{1+\varepsilon^2}}$

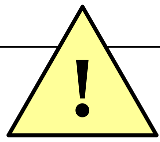
- Most efficient (η)
  - the **largest ratio** of the passband gain to stopband gain
  - **or** for a given ratio of passband to stopband gain, it requires the **smallest transition band**

➔ **in MATLAB:** `elliporд` followed by `ellip`

In Summary

Filter Type	Passband Ripple	Stopband Ripple	Transition Band	MATLAB Design Command
Butterworth	No	No	Loose	`butter`
Chebyshev	Yes	No	Tight	`cheby`
Chebyshev Type II (Inverse Chebyshev)	No	Yes	Tight	`cheby2`
Eliptic	Yes	Yes	Tightest	`ellip`

Announcements:



- Assignment 1 Solutions:
  - Extended to Today!
  - Solution for Problem 7 Posted
  - "Best of" Submissions
    - Did you see / have a "best of" submission – let us know about it
    - We'll post these to the class webpage (with permission of the student of course)
- Lab 3 (Experiment 4)
  - **<u>Will run on Week 9!</u>**
  - ∵ Week 8 has the ANZAC holiday

Next Time in Linear Systems ….

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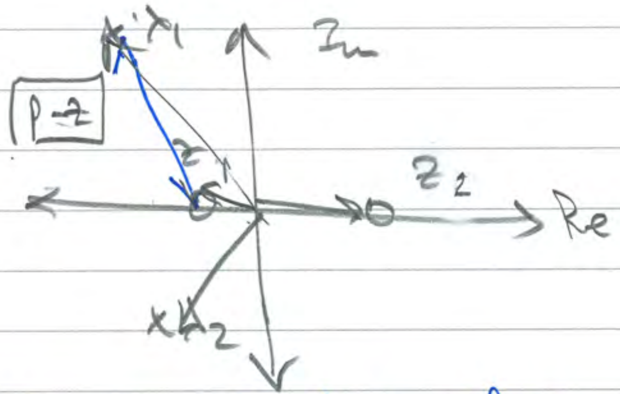
→ FILTER DESIGN BY POLE/ZERO PLACEMENT

$$H(s) = \frac{P(s)}{Q(s)} = b_n \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)}$$

z: ZEROS
λ: POLES



→ BUTTERWORTH FILTERS IN POLE/ZERO NOTATION

$$H(j\omega) = \frac{1}{\sqrt{1+\omega^{2n}}}$$

Normalized form: $\omega_c = 1$

$s = j\omega$

$$H(s) = \frac{1}{\sqrt{1+\left(\frac{s}{j}\right)^{2n}}}$$

Why: $\omega = \frac{s}{j}$

$$H(s)^2 = \frac{1}{1+\left(\frac{s}{j}\right)^{2n}}$$

$$|H(s)|^2 = \frac{1}{1+(s/j)^{2n}} = \frac{1}{1+\frac{s^{2n}}{j^{2n}}}$$

POLES ARE GIVEN BY:

poles at: $s^{2n} = -(j^{2n})$

why? $\because \left(1+\frac{s^{2n}}{j^{2n}}\right) = 0 \Rightarrow \frac{s^{2n}}{j^{2n}} = -1$

$\therefore s^{2n} = -(j^{2n})$

Recall that:

$$-1 = e^{j\pi(2k-1)}$$

$$e^{j\pi/2} = j$$

$$\Rightarrow s^{2n} = e^{j\pi(2k-1+n)}$$

k = integer

→ The poles can be visualized on a pole-zero Diagram: (Ex. 7.22)

→ OBSERVE that all poles have a unit magnitude and are located on a unit circle in the s-plane separated by an angle

$$\frac{\pi}{n}$$

→ Butterworth Transfer function

$$H(s) = \frac{1}{B(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + 1}$$

where: $a_{n-1}, \ldots, a_1$ = from Butterworth tables