Noise & Filters

ELEC 3004: **Systems**: Signals & Controls
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Lecture 12

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Goals for the Week

- Continue of Discussion of the z-Transform
- Noise
- Introduce Filters (Today) ➔ Analog Filters Friday
  - Frequency Response of an LTIC and LTID System
  - Butterworth
  - Chebyshev
- Digital Filters IIR Filters
    - Design of IIR Filters from Analog Filters
  - FIR Filter

The z-Transform

- It is defined by:

$$z = re^{j\omega}$$

Or in the Laplace domain:

$$z = e^{sT}$$

- Thus: $Y(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ or $y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$

- I.E., It's a discrete version of the Laplace:

$$f(kT) = e^{-akT} \Rightarrow \mathcal{Z}\{f(k)\} = \frac{z}{z - e^{-aT}}$$

The *z*-transform

- In practice, you'll use look-up tables or computer tools (ie. Matlab) to find the *z*-transform of your functions

$F(s)$	$F(kt)$	$F(z)$
$\frac{1}{s}$	1	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\frac{1}{s^2+a^2}$	$\sin(akT)$	$\frac{z \sin aT}{z^2-(2\cos aT)z+1}$

An example!

- Back to our difference equation:

$$y(k) = x(k) + Ax(k-1) - By(k-1)$$

becomes

$$Y(z) = X(z) + Az^{-1}X(z) - Bz^{-1}Y(z)$$
$$(z+B)Y(z) = (z+A)X(z)$$

which yields the transfer function:

$$\frac{Y(z)}{X(z)} = \frac{z+A}{z+B}$$

Note: It is also not uncommon to see systems expressed as polynomials in z^{-n}

This looks familiar…

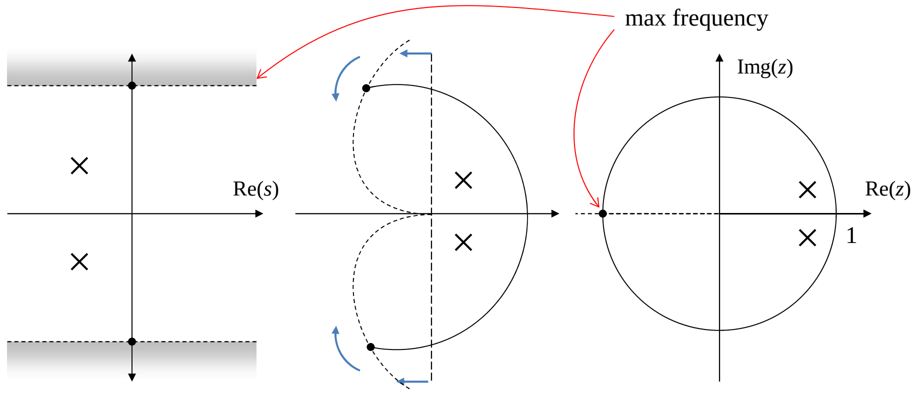
- Compare:

$$\frac{Y(s)}{X(s)} = \frac{s+2}{s+1} \text{ vs } \frac{Y(z)}{X(z)} = \frac{z+A}{z+B}$$

How are the Laplace and *z* domain representations related?

Deep insight #1

The mapping between continuous and discrete poles and zeros acts like a distortion of the plane



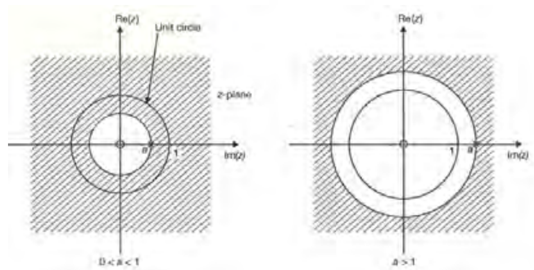
Region of Convergence

- For the convergence of X(z) we require that

$$\sum_{n=0}^{\infty} \left| az^{-1} \right|^{n} < \infty$$

- Thus, the ROC is the range of values of z for which $|az^{-1}| < 1$ or, equivalently, $|z| > |a|$. Then

$$X(z) = \frac{z}{z-a} \qquad |z| > |a|$$



Z-Transform Properties: Linearity

Linearity:

$$a_1 y_1[n] + a_2 y_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} a_1 Y_1(z) + a_2 Y_2(z)$$

Z-Transform Properties: Time Shifting

$$y[n - n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} Y(z)$$

$$\begin{aligned} y_2[n] &= y[n - n_0] \\ Y_2(e^{jw}) &= \sum_{k=-\infty}^{\infty} y[k - n_0] z^{-k} \\ &= \sum_{l=-\infty}^{\infty} y[l] z^{-(l+n_0)} \\ &= z^{-n_0} Y(z) \end{aligned}$$

- Two Special Cases:
- z^{-1}: the *unit-delay operator:*

$$x[n-1] \leftrightarrow z^{-1} X(z) \qquad R' = R \cap \{0 < |z|\}$$

- z: *unit-advance operator:*

$$x[n+1] \leftrightarrow z X(z) \qquad R' = R \cap \{|z| < \infty\}$$

Z-Transform Properties

- Time Reversal

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \qquad R' = \frac{1}{R}$$

- Multiplication by z^n

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \qquad R' = |z_0| R$$

- Multiplication by n (or Differentiation in z):

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \qquad R' = R$$

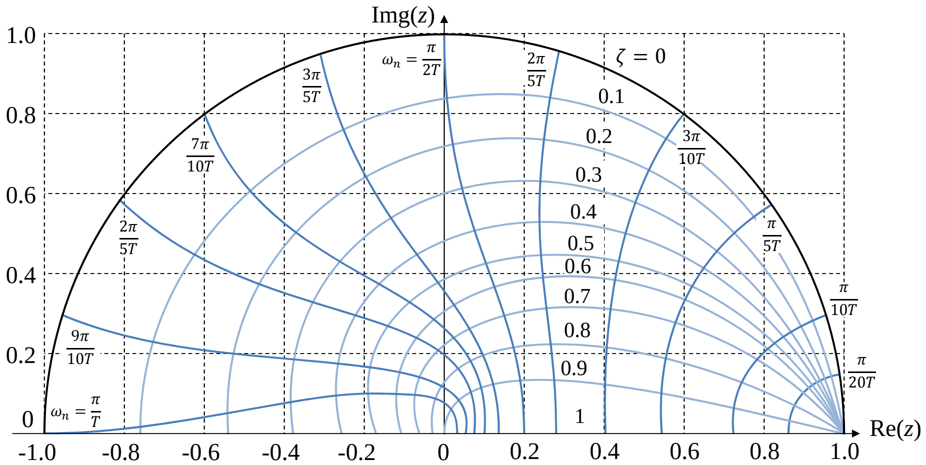
- Convolution

$$x_1[n] \leftrightarrow X_1(z) \qquad \text{ROC} = R_1$$
$$x_2[n] \leftrightarrow X_2(z) \qquad \text{ROC} = R_2$$

$$x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \qquad R' \supset R_1 \cap R_2$$

z-transform & Systems: Damping and natural frequency

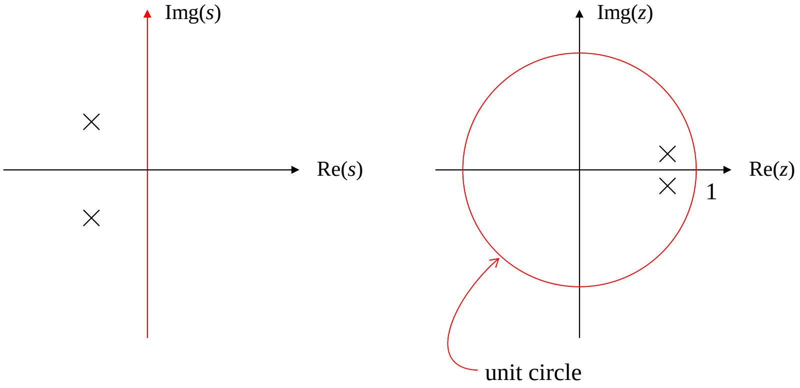
$$z = e^{sT} \text{ where } s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



[Adapted from Franklin, Powell and Emami-Naeini]

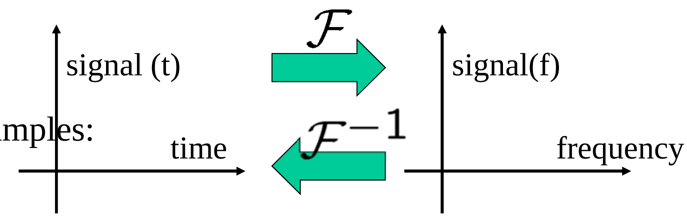
In general: *z*-plane stability

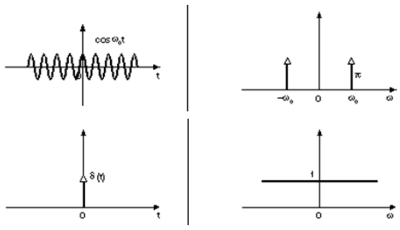
- In the *z*-domain, the unit circle is the system stability bound



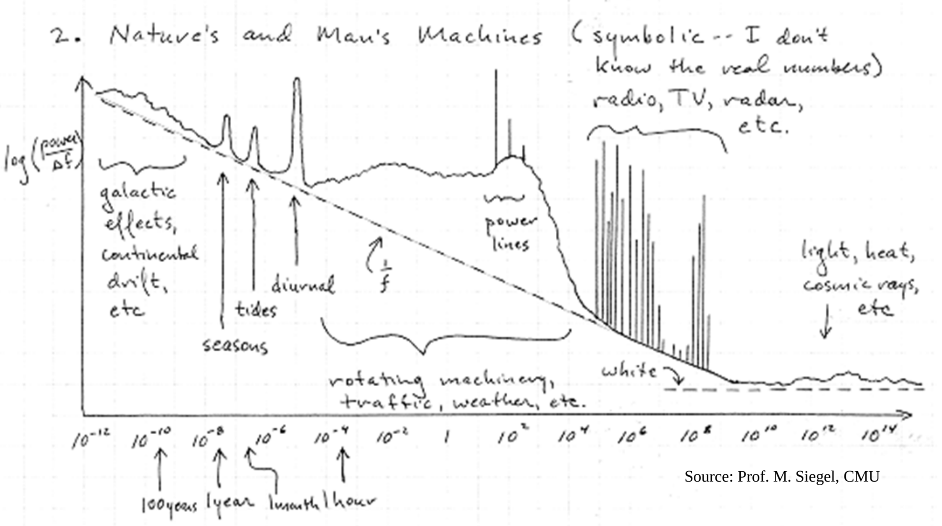
Frequency

- How often the signal repeats
- Can be analyzed through Fourier Transform
- Examples:





Noise



Source: Prof. M. Siegel, CMU

Note: this picture illustrates the concepts but it is not quantitatively precise

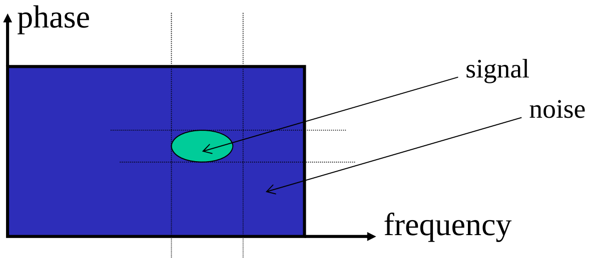
Noise [2]

Various Types:

- Thermal (white):
  - Johnson noise, from thermal energy inherent in mass.
- Flicker or 1/f noise:
  - Pink noise
  - More noise at lower frequency
- Shot noise:
  - Noise from quantum effects as current flows across a semiconductor barrier
- Avalanche noise:
  - Noise from junction at breakdown (circuit at discharge)

How to beat the noise

- Filtering (Narrow-banding): Only look at particular portion of **frequency space**
- Multiple measurements …
- Other (modulation, etc.) …



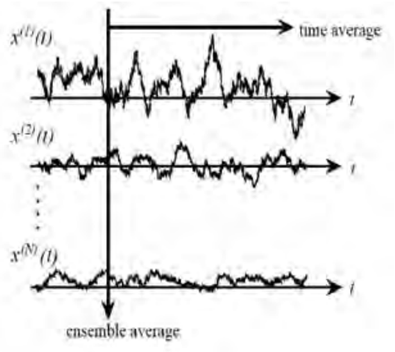
Noise ⊆ Uncertainty

- **Uncertainty**:
  All measurement has some approximation
  - A. Statistical uncertainty: quantified by mean & variance
  - B. Systematic uncertainty: non-random error sources

- **Law of Propagation of Uncertainty**
  - Combined uncertainty is root squared

$$u_c = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

Treating Uncertainty with Multiple Measurements



1. **Over time:** multiple readings of a quantity over time
   - "stationary" or "ergodic" system
   - Sometimes called "integrating"

2. **Over space:** single measurement (summed) from multiple sensors each distributed in space

3. **Same Measurand**: multiple measurements take of the **same observable quantity** by multiple, related instruments

   e.g., measure position & velocity simultaneously

→ Basic "sensor fusion"

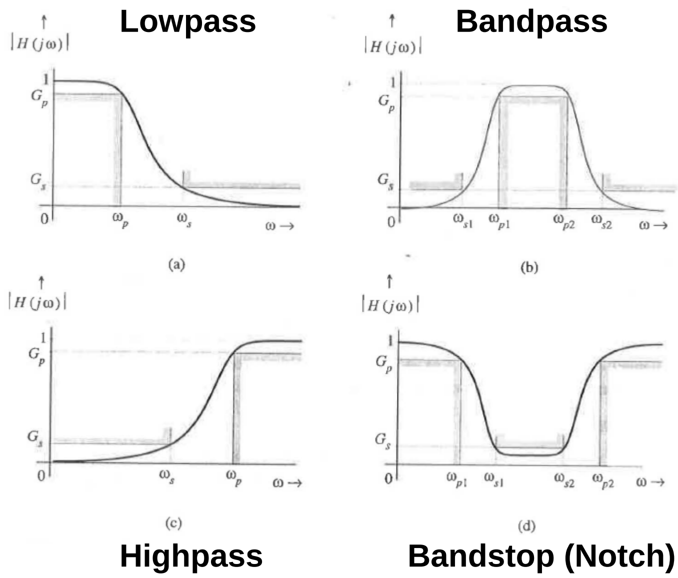
.

$$\sigma_{\text{final}} = \left[\sigma_1^{-1} + \sigma_2^{-1} + \cdots + \sigma_n^{-1}\right]^{-1}$$

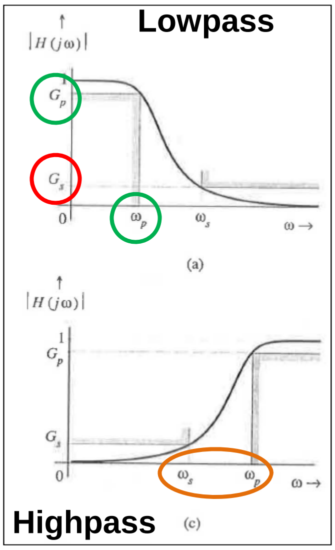
Multiple Measurements Example

- What time was it when this picture was taken?
- What was the temperature in the room?

Filters



Filters



Specified Values:

- Gp = minimum passband gain

Typically:

$$G_p = \frac{1}{\sqrt{2}} = -3dB$$

- Gs = maximum stopband gain
  - **Low**, not zero (sorry!)
  - For realizable filters, the gain cannot be zero over a finite band (Paley-Wiener condition)
- Transition Band:

  transition from the passband to the stopband ➔ $\omega p \neq \omega s$

Announcements:

- Assignment 1 Solutions:
  - Extended to Friday

- Lab 2 (Experiment 3)

  ➔We are rerunning it again this week ☺
  (for those who want to do it "post-theory")

- Lab 3 (Experiment 4)
  - **<u>Will run on Week 9!</u>**

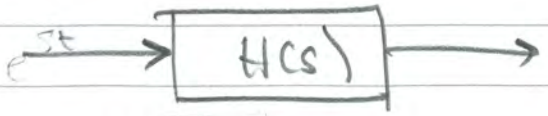
  ∵ Week 8 has the ANZAC holiday

Next Time in Linear Systems ….

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Signals & Signal Models
	8-Mar	System Models
3	13-Mar	Linear Dynamical Systems
	15-Mar	Sampling & Data Acquisition
4	20-Mar	Time Domain Analysis of Continuous Time Systems
	22-Mar	System Behaviour & Stability
5	27-Mar	Signal Representation
	29-Mar	Holiday
6	10-Apr	Frequency Response
	12-Apr	z-Transform
7	17-Apr	Noise & Filtering
	19-Apr	Analog Filters
8	24-Apr	Discrete-Time Signals
	26-Apr	Discrete-Time Systems
9	1-May	Digital Filters & IIR/FIR Systems
	3-May	Fourier Transform & DTFT
10	8-May	State-Space
	10-May	Controllability & Observability
11	15-May	Introduction to Digital Control
	17-May	Stability of Digital Systems
12	22-May	PID & Computer Control
	24-May	Information Theory & Communications
13	29-May	Applications in Industry
	31-May	Summary and Course Review

→ Frequency Response of a LTIC System

system:



input: $U(s)$

output: $Y(s) = H(s) \cdot U(s)$

→ Example: $H(s) = \dfrac{s+0.1}{s+0.5} = \dfrac{s+0.1}{s+5}$

→ $s = j\omega$

$$H(j\omega) = \frac{j\omega + 0.1}{j\omega + 5}$$

$$|H(j\omega)| = \frac{\sqrt{\omega^2 + 0.01}}{\sqrt{\omega^2 + 25}}$$

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$$

why?

$$H(j\omega) = \frac{j\omega + A}{j\omega + B}$$

$$H(j\omega) = \frac{\sqrt{\omega^2 + A^2}}{\sqrt{\omega^2 + B^2}}$$

$$\angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{A}\right) - \tan^{-1}\left(\frac{\omega}{B}\right)$$

→ Example 2: Time Delay

$$H(s) = e^{-sT}$$

$$H(j\omega) = e^{-j\omega T}$$

$$|H(j\omega)| = 1 \quad \text{and} \quad \angle H(j\omega) = -\omega T$$

→ FILTERS

→ COLORED SIGNAL