Frequency Response

ELEC 3004: **Systems**: Signals & Controls
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Lecture 10

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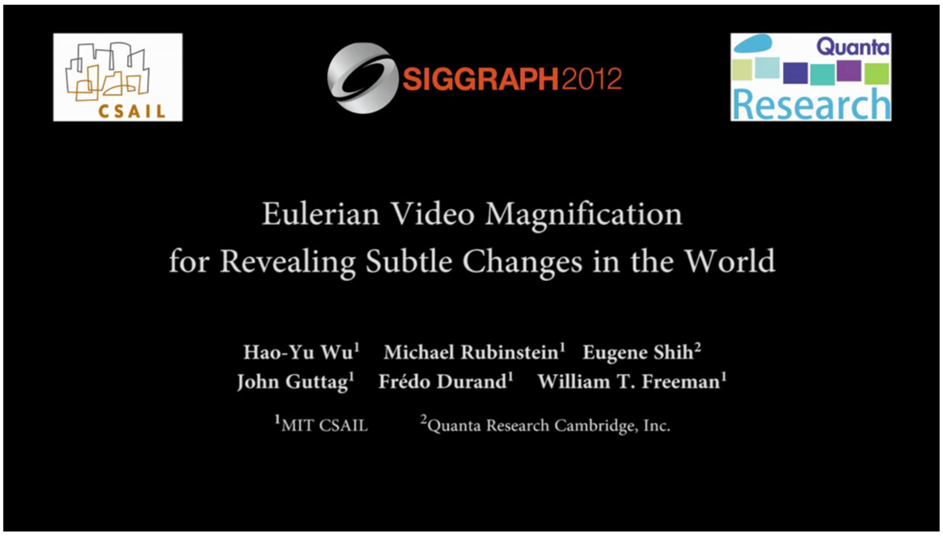
April 10, 2013

Today:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Signals & Signal Models
	8-Mar	System Models
3	13-Mar	Linear Dynamical Systems
	15-Mar	Sampling & Data Acquisition
4	20-Mar	Time Domain Analysis of Continuous Time Systems
	22-Mar	System Behaviour & Stability
5	27-Mar	Signal Representation
	29-Mar	Holiday
6	10-Apr	Frequency Response & Fourier Transform
	12-Apr	Analog Filters
7	17-Apr	IIR Systems
	19-Apr	FIR Systems
8	24-Apr	z-Transform
	26-Apr	Discrete-Time Signals
9	1-May	Discrete-Time Systems
	3-May	Digital Filters
10	8-May	State-Space
	10-May	Controllability & Observability
11	15-May	Introduction to Digital Control
	17-May	Stability of Digital Systems
12	22-May	PID & Computer Control
	24-May	Information Theory & Communications
13	29-May	Applications in Industry
	31-May	Summary and Course Review

Cool Signal Share: Eulerian Video Magnification for Revealing Subtle Changes in the World

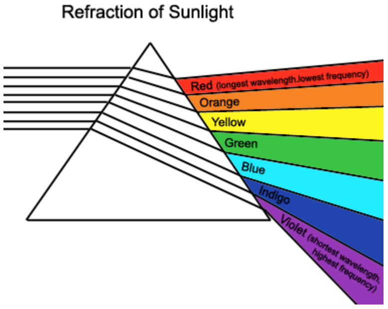


Announcements:

- Assignment 1 Solutions:
  - Posted
  - Please try to get the peer review marks in by **<u>April 16</u>** at 11:59pm

Signals Processing: Seeing The Light

White light
(all wavelengths)



Colour
spectrum

f= 1/λ

Think of a Signal Processing like a prism:
"Destructs a source signal into its constituent frequencies"

Fourier Series

- Any finite power, periodic, signal x(t)
  - period T
- can be represented as (∞) summation of
  - sine and cosine waves
- Called: Trigonometrical Fourier Series

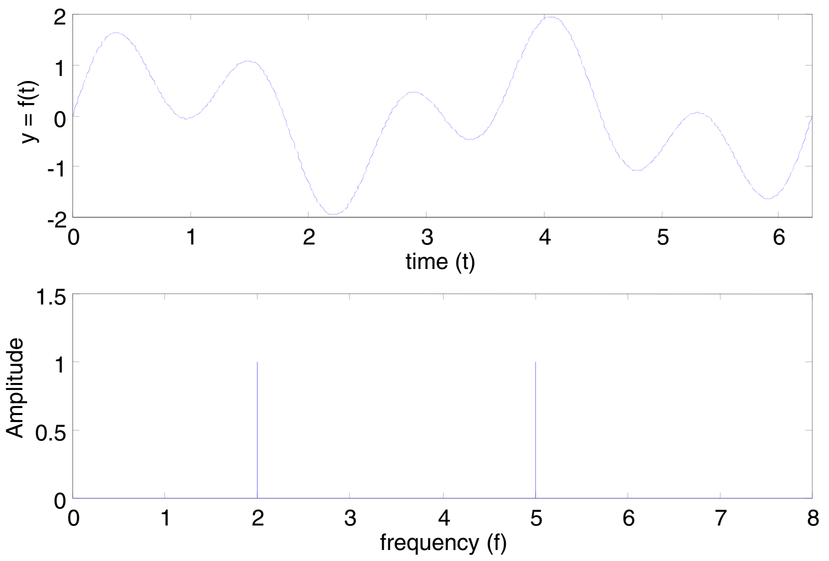
$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nw_0 t) + B_n \sin(nw_0 t)$$

- Fundamental frequency $w_0=2\pi/T$ rad/s or $1/T$ Hz
- DC (average) value $A_0/2$

Transform Analysis

- Signal measured (or known) as a function of an independent variable
  - e.g., time: y = f(t)
- However, this independent variable may not be the most appropriate/informative
  - e.g., frequency: Y = f(w)
- Therefore, need to transform from one domain to the other
  - e.g., time ⇔ frequency
  - As used by the human ear (and eye)

Signal processing uses Fourier, Laplace, & *z* transforms etc



Frequency representation (spectrum) shows signal contains:
• 2Hz and 5Hz components (sinewaves) of equal amplitude

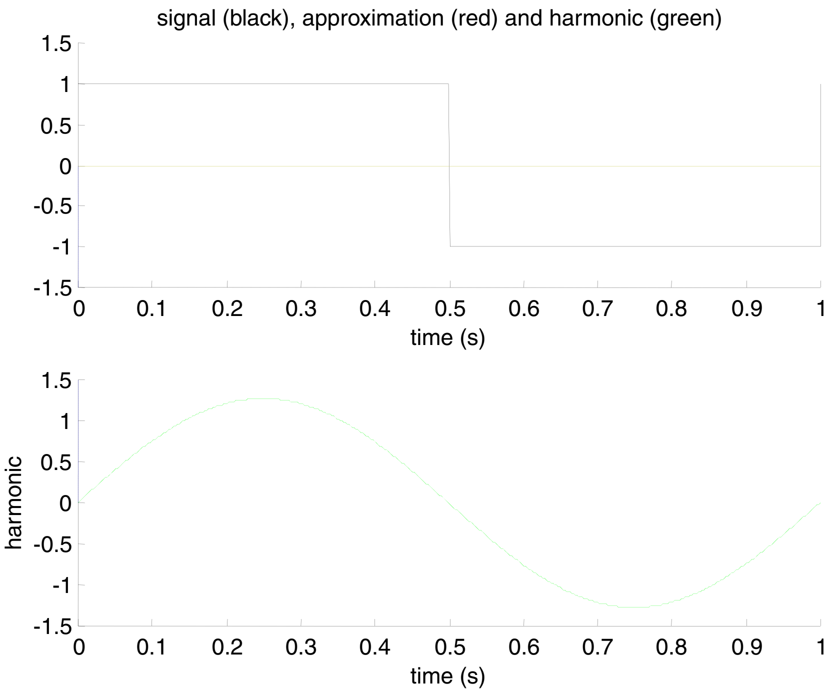
Fourier Series Coefficients

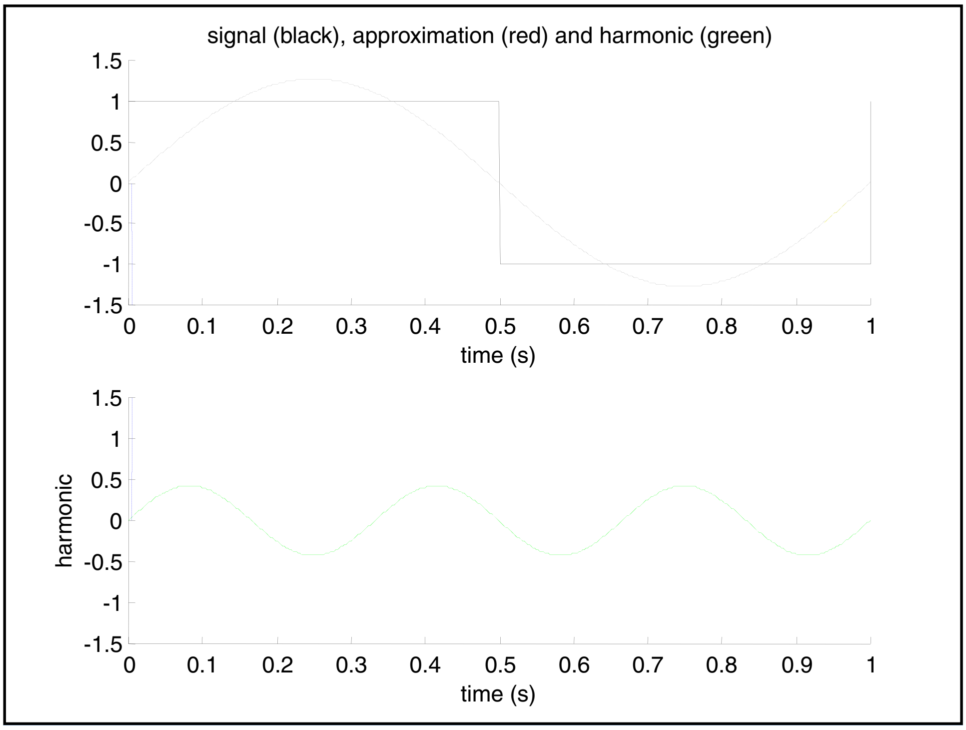
- An & Bn calculated from the signal, x(t)
  - called: Fourier coefficients

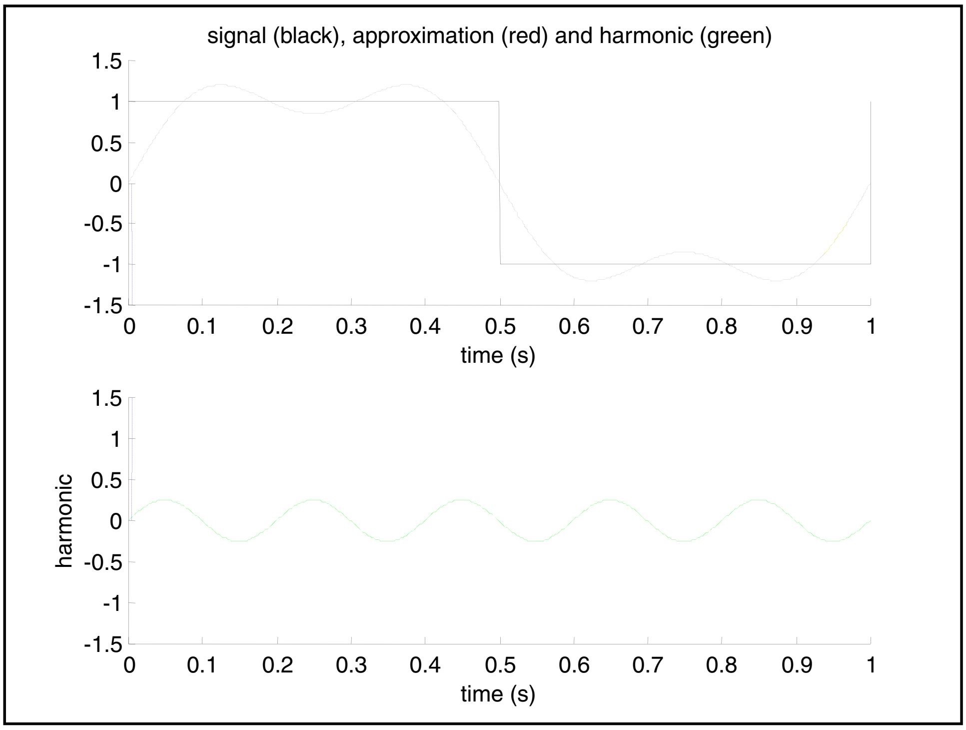
$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(nw_0 t)dt \qquad n = 0,1,2,\cdots$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(nw_0 t)dt \qquad n = 1,2,3,\cdots$$

Note: Limits of integration can vary, provided they cover one period




signal (black), approximation (red) and harmonic (green)
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time (s)
harmonic
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time (s)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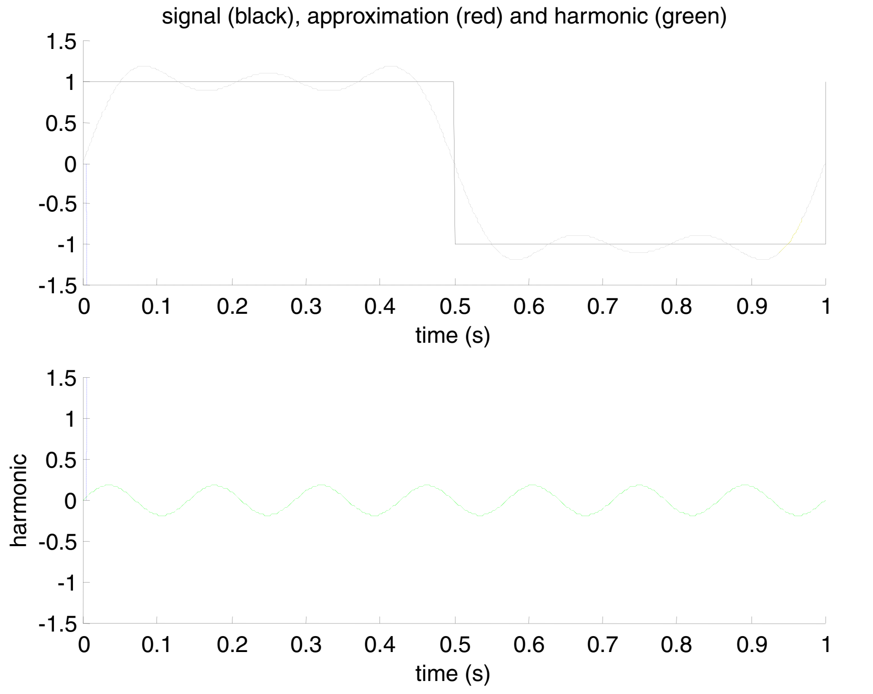
0.9
1
time (s)



Fourier Series Coefficients

- Approximation with 1st, 3rd, 5th, & 7th Harmonics added, note:
  - 'Ringing' on edges due to series truncation
  - Often referred to as Gibb's phenomenon
- Fourier series converges to original signal if
  - Dirichlet conditions satisfied
  - Closer approximation with more harmonics

Dirichlet Conditions

- For Fourier series to converge, f(t) must be
- defined & single valued
- continuous and have a finite number of finite discontinuities within a periodic interval, and
- piecewise continuous in periodic interval, as must f'(t)

Example: Square wave

$$x(t) = \begin{cases} 1, & 0 < t < 1; \\ -1, & 1 < t < 2; \\ x(t+2). & \end{cases}$$

← periodic! i.e., x(t + 2) = x(t)

$$A_n = \int_0^2 x(t)\cos(n\pi t)dt = \int_0^1 \cos(n\pi t)dt - \int_1^2 \cos(n\pi t)dt$$

$$A_n = \left[\frac{-\sin(n\pi t)}{n\pi}\right]_0^1 - \left[\frac{-\sin(n\pi t)}{n\pi}\right]_1^2 = 0$$

No cos terms as $\sin(n\pi) = 0 \; \forall \; n$
x(t) has odd symmetry

$$B_n = \int_0^2 x(t)\sin(n\pi t)dt = \int_0^1 \sin(n\pi t)dt - \int_1^2 \sin(n\pi t)dt$$

$\cos(2n\pi) = 1 \; \forall \; n$

$$B_n = \left[\frac{-\cos(n\pi t)}{n\pi}\right]_0^1 - \left[\frac{-\cos(n\pi t)}{n\pi}\right]_1^2 = -\frac{\cos(n\pi)}{n\pi} + \frac{1}{n\pi} + \frac{1}{n\pi} - \frac{\cos(n\pi)}{n\pi}$$

$$B_n = \frac{2}{n\pi}(1 - \cos(n\pi))$$

Sin terms only

Example: Square wave

Therefore, Trigonometric Fourier series is,

$$x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-\cos(n\pi))\sin(n\pi t)$$

Expanding the terms gives,

$$\begin{aligned} x(t) = \; & \frac{4}{\pi}\sin(\pi t) && \text{(fundamental)} \\ & +0 && \text{(second harmonic)} \\ & +\frac{4}{3\pi}\sin(3\pi t) && \text{(third harmonic)} \\ & +0 && \text{(fourth harmonic)} \\ & +\frac{4}{5\pi}\sin(5\pi t) && \text{(fifth harmonic)} \\ & +\text{etc} \end{aligned}$$

- Only odd harmonics;
- In proportion 1,1/3,1/5,1/7,…
- Higher harmonics contribute less;
- Therefore, converges

Complex Fourier Series (CFS)

- Also called Exponential Fourier series
  - As it uses Euler's relation
- FS as a Complex phasor summation

$$A\exp(jw_0t) = A\cos(w_0t) + jA\sin(w_0t)$$

which implies,

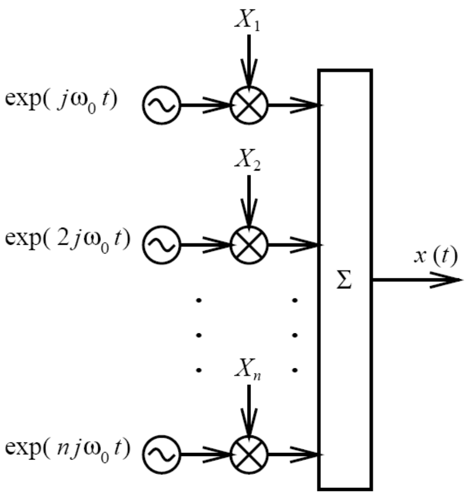
$$\cos(nw_0t) = \frac{\exp(jnw_0t)+\exp(-jnw_0t)}{2}$$

$$\sin(nw_0t) = \frac{\exp(jnw_0t)-\exp(-jnw_0t)}{2j}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jnw_0t)$$

Where X_n are the CFS coefficients

Complex Phasor Summation



Complex Fourier Coefficients

- Again, Xn calculated from x(t)
- Only one set of coefficients, Xn
  - but, generally they are complex

$$X_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\exp(-jnw_0 t)dt$$

Remember: fundamental $w_0 = 2\pi/T$!

Relationships

- There is a simple relationship between
  - trigonometrical and
  - complex Fourier coefficients,

$$X_0 = \frac{A_0}{2}$$

$$X_n = \begin{cases} \dfrac{A_n - jB_n}{2}, & n > 0; \\ \dfrac{A_n + jB_n}{2}, & n < 0. \end{cases}$$

Constrained to be symmetrical, i.e., complex conjugate

$$X_{-n} = X_n^*$$

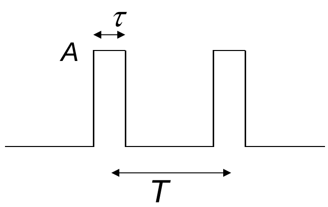
Therefore, can calculate simplest form and convert

Example: Complex FS

- Consider the pulse train signal

$$x(t) = \begin{cases} A, & 0 \le |t| \le \dfrac{\tau}{2}; \\ 0, & \dfrac{\tau}{2} < |t| \le T; \\ x(t+T). & \end{cases}$$

- Has complex Fourier series:



$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\exp(-jn\omega_0 t)dt$$

$$= \frac{-A\tau}{jn\omega_0 T\tau}\left[\exp\left(\frac{-jn\omega_0\tau}{2}\right) - \exp\left(\frac{jn\omega_0\tau}{2}\right)\right]$$

Note: × by τ/τ …

Note: *n* is the ind. variable

Example: Complex FS

- Which using Euler's identity reduces to:

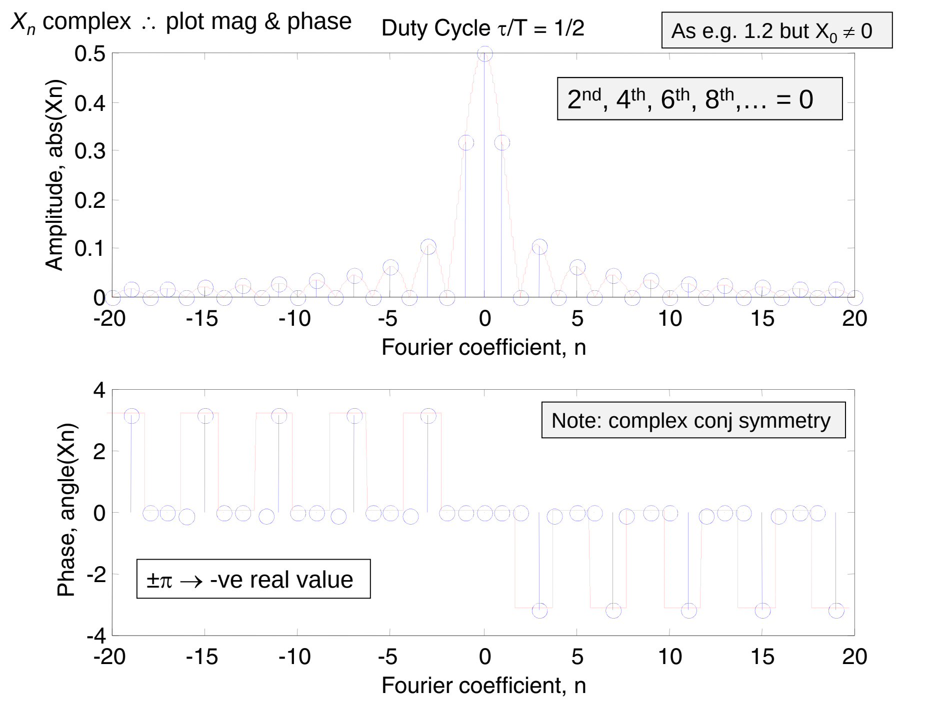
$$X_n = \frac{A\tau}{T}\frac{\sin(nw_0\,\tau/2)}{nw_0\,\tau/2} = \frac{A\tau}{T}\mathrm{sa}(nw_0\,\tau/2)$$

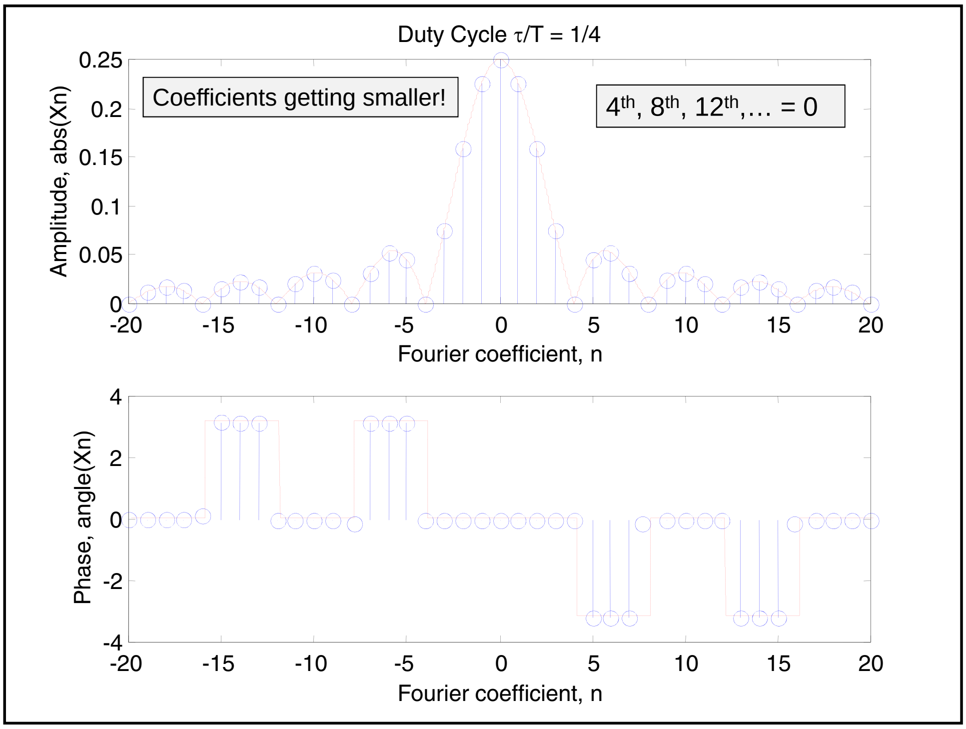
$$w_0 = \frac{2\pi}{T}$$

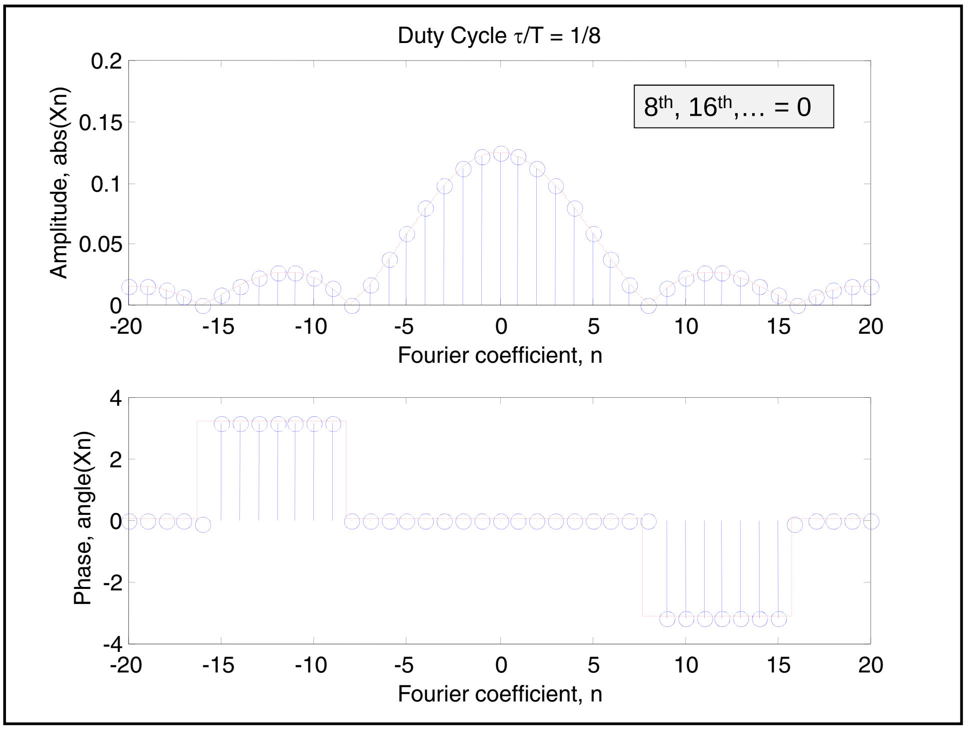
Note: letting $\theta = \frac{n\omega_0\tau}{2}$

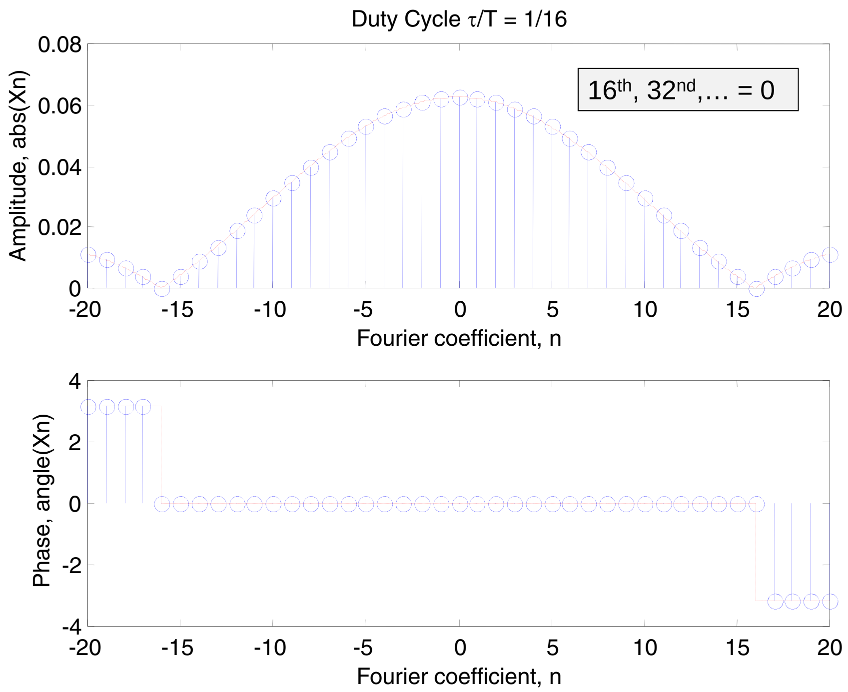
$$\exp(-j\theta) - \exp(j\theta)$$
$$= \cos(-\theta) + j\sin(-\theta) - (\cos(\theta) + j\sin(\theta))$$
$$= \cos(\theta) - j\sin(\theta) - \cos(\theta) - j\sin(\theta) \quad = \quad -2j\sin(\theta)$$

Note:
cos(-θ) = cos(θ): even
sin(-θ) = -sin(θ): odd




Duty Cycle τ/T = 1/4
Coefficients getting smaller!
4th, 8th, 12th,… = 0
Amplitude, abs(Xn)
Fourier coefficient, n
Phase, angle(Xn)
Fourier coefficient, n


Duty Cycle τ/T = 1/8
8th, 16th,… = 0
Amplitude, abs(Xn)
Fourier coefficient, n
Phase, angle(Xn)
Fourier coefficient, n



Complex FS of Pulse Train

- Amplitude spectrum has 'sinc' ('sa') envelope
  - Amplitude reduces as duty cycle decreases
  - DC coefficient X0 (n=0) present
    - compare to first example
  - Duty cycle τ/T: XMτ/T = 0 (M = 1,2,3,…)
  - Even symmetry: |X-n| = |Xn|
    - For real (not complex) signals (all we shall consider)
    - Often only plot positive frequency, e.g., Matlab
- Phase spectrum
  - Odd symmetry: ∠ X-n = -∠ Xn

Complex conjugate (Hermitian) symmetry is general property for real x(t)

Interpretation of Fourier Series

- Represents periodic signals (T = 2π/w0)
  - as sum of cosine waves: "cosine series"
    - at harmonic frequencies 0, w0, 2w0, 3w0, …
    - |Xn| is half amplitude of nth harmonic
    - ∠Xn is phase shift of nth harmonic
- Distribution with harmonic number of
  - both amplitude & phase
  - called a frequency spectrum (discrete)

$$x(t) = X_0 + \sum_{n=1}^{\infty} 2\,|X_n|\cos\left(nw_0 t + \angle X_n\right)$$

i.e., Harmonics only

Orthogonal Expansions

- Trigonometrical and Complex FS both
  - Orthogonal expansions
- Because harmonically related
  - sines, cosines, and complex phasors are all orthogonal
    - Product of f1(t) & f2(t) integrates to zero over 1 period

$$\frac{1}{T}\int_{-T/2}^{T/2} \exp(jnw_0 t)\exp^*(jmw_0 t)\,dt = \begin{cases} 1, & \text{if } m = n; \\ 0, & \text{if } m \neq n. \end{cases}$$

This means we can calculate each X_n independently
(instead of solving *n* simultaneous equations)

Example: Calculate Power in x(t)

A simple periodic signal $x(t)$ where:

$$x(t) = a_1 \sin(\omega_0 t) + a_2 \sin(2\omega_0 t)$$

The power:

$$\begin{aligned} P &= \frac{1}{T}\int_0^T x^2(t)\,dt \\ &= \frac{1}{T}\int_0^T a_1^2 \sin^2(\omega_0 t)\,dt \\ &\quad + \frac{2}{T}\int_0^T a_1\,a_2 \sin(\omega_0 t)\sin(2\omega_0 t)\,dt \\ &\quad + \frac{1}{T}\int_0^T a_2^2 \sin^2(2\omega_0 t)\,dt \\ &= \frac{1}{T}\int_0^T a_1^2 \sin^2(\omega_0 t)\,dt + \frac{1}{T}\int_0^T a_2^2 \sin^2(2\omega_0 t)\,dt \\ &= \frac{a_1^2}{2} + \frac{a_2^2}{2} \end{aligned}$$

Orthogonal $\rightarrow$ 0

Total = sum of power in 2 sine waves

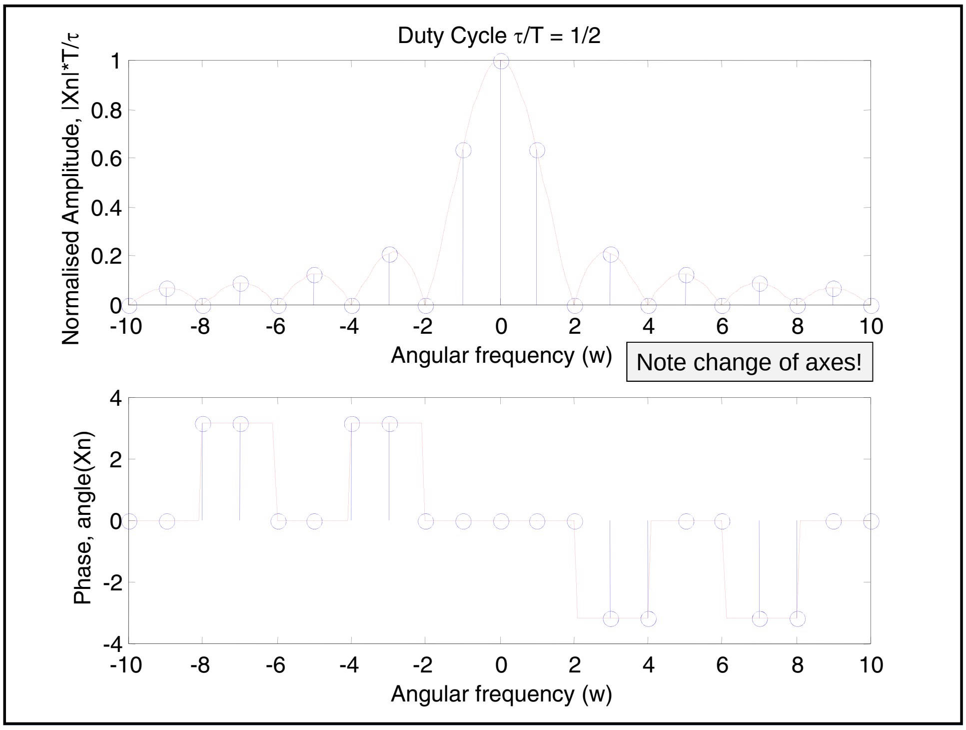
Parseval's Theorem

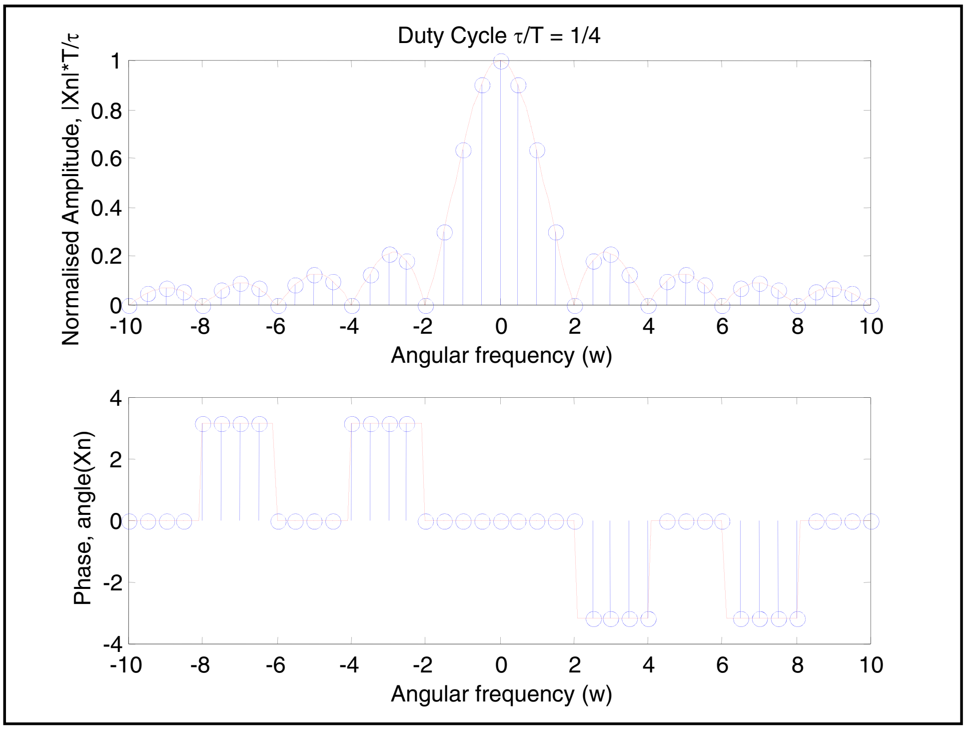
- Direct consequence of orthogonality
- Calculate power of a signal either
  - in time domain, i.e., from x(t), or
  - in frequency domain, i.e., from Xn

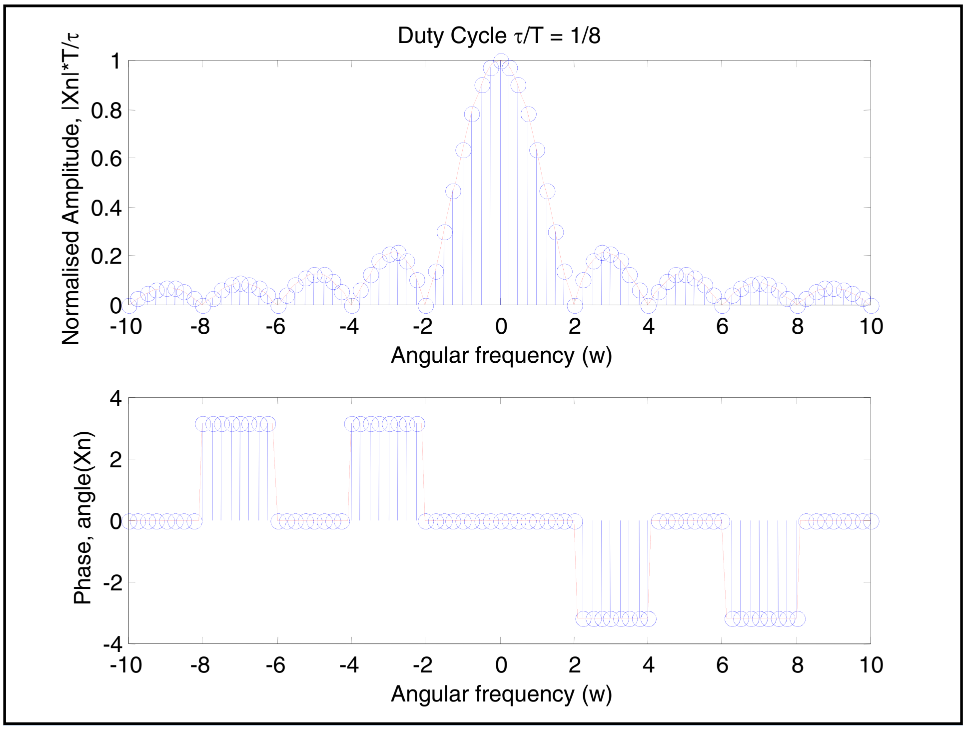
$$P = \frac{1}{T}\int_0^T x^2(t)dt = \sum_{n=-\infty}^{\infty} |X_n|^2$$

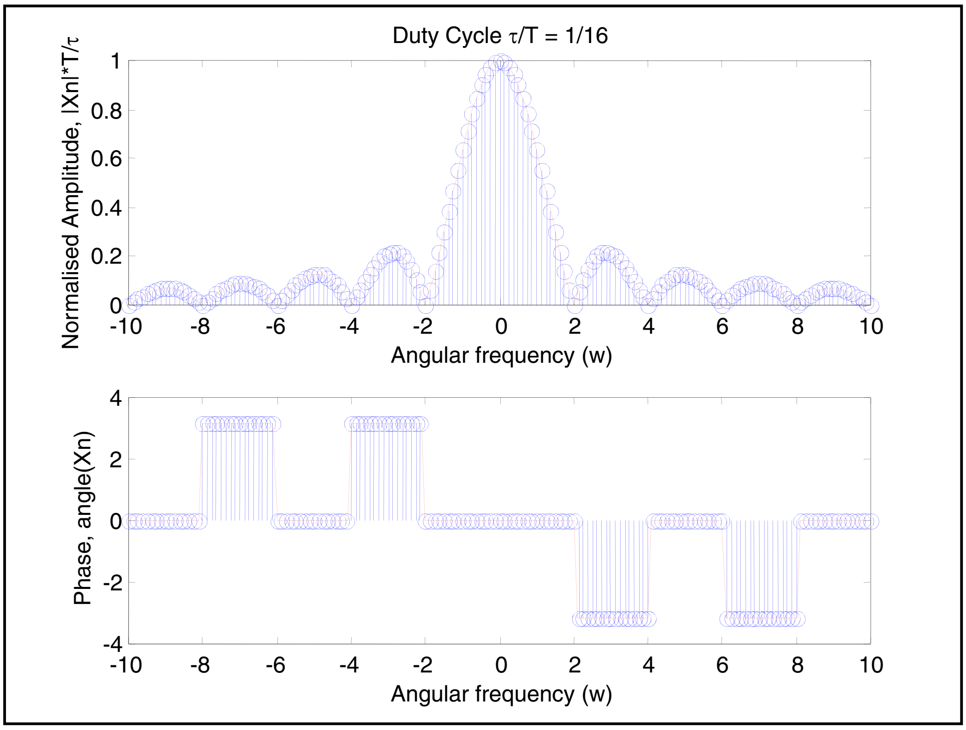
Continuous & periodic

Discrete


Duty Cycle τ/T = 1/2
Normalised Amplitude, |Xn|*T/τ
Angular frequency (w)
Note change of axes!
Phase, angle(Xn)
Angular frequency (w)


Duty Cycle τ/T = 1/4
Normalised Amplitude, |Xn|*T/τ
Angular frequency (w)
Phase, angle(Xn)
Angular frequency (w)


Duty Cycle τ/T = 1/8
Normalised Amplitude, |Xn|*T/τ
Angular frequency (w)
Phase, angle(Xn)
Angular frequency (w)


Duty Cycle τ/T = 1/16
Normalised Amplitude, |Xn|*T/τ
Angular frequency (w)
Phase, angle(Xn)
Angular frequency (w)

Fourier Transform

- Fourier series
  - Only applicable to periodic signals
- Real world signals are rarely periodic
- Develop Fourier transform by
  - Examining a periodic signal
  - Extending the period to infinity

Fourier Transform

- Problem: as $T \to \infty$, $Xn \to 0$
  - i.e., Fourier coefficients vanish!
- Solution: re-define coefficients
  - $Xn' = T \times Xn$
- As $T \to \infty$
  - (harmonic frequency) nw0 $\to$ w (continuous freq.)
  - (discrete spectrum) Xn' $\to$ X(w) (continuous spect.)
  - w0 (fundamental freq.) reduces $\to$ dw (differential)
    - Summation becomes integration

Fourier Transform

Note missing 1/T

$$X_n^{'} = \int_{-T/2}^{+T/2} x(t)\exp(-jnw_0 t)dt$$

Modified Fourier series

$$X(w) = \lim_{T\to\infty} \int_{-T/2}^{+T/2} x(t)\exp(-jnw_0 t)dt$$

Note: as T $\to \infty$
$n\omega_0 \to \omega$

$$X(w) = \int_{-\infty}^{\infty} x(t)\exp(-jwt)dt$$

Fourier transform
X(w) = F{x(t)}

$$x(t) = \lim_{T\to\infty} \sum_{-\infty}^{\infty} X_n^{'} \exp(jnw_0 t)\frac{w_0}{2\pi}$$

Inverse Fourier series

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\exp(jwt)dw$$

Inverse Fourier transform
x(t) = F^{-1}{X(w)}

As a example of the evaluation of the Fourier transform, consider the finite energy signal $x(t)$ illustrated in Figure 1.8.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\,dt$$
$$= \int_{-\tau/2}^{\tau/2} x(t)\exp(-j\omega t)\,dt$$
$$= \frac{-A}{j\omega}\left[\exp\left(\frac{-j\omega\tau}{2}\right) - \exp\left(\frac{j\omega\tau}{2}\right)\right]$$
$$= A\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}$$
$$= A\tau\ \text{sinc}(\omega\tau/2)$$

Using Euler's relation

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Note the similarity to Previous FS examples

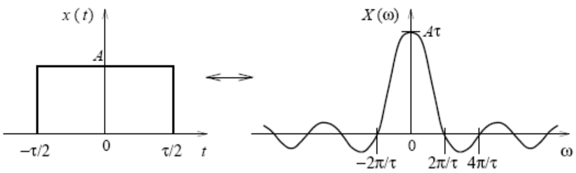


Figure 1.8 *Fourier transform of a rectangular pulse.*

Summary!