Question 1

We know that every signal x(t) can be expressed as a sum of even and odd components (As seen in Lathi section 1.5.2):

$x(t) = x_e(t) + x_o(t)$

$x(t) = \frac{1}{2}[x(t) + x(-t)] + \frac{1}{2}[x(t) - x(-t)]$

where,

$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

Given that $x(t) = sin(\pi t)u(t)$, the odd component of signal represented as $x_o(t)$ can be further expressed as:

$x_o(t) = \frac{1}{2}[sin(\pi t)u(t) - sin(-\pi t)u(-t)]$

Using the identity $sin(-t) = -sin(t)$ and $u(-t) = 1 - u(t)$ this simplifies as follows:

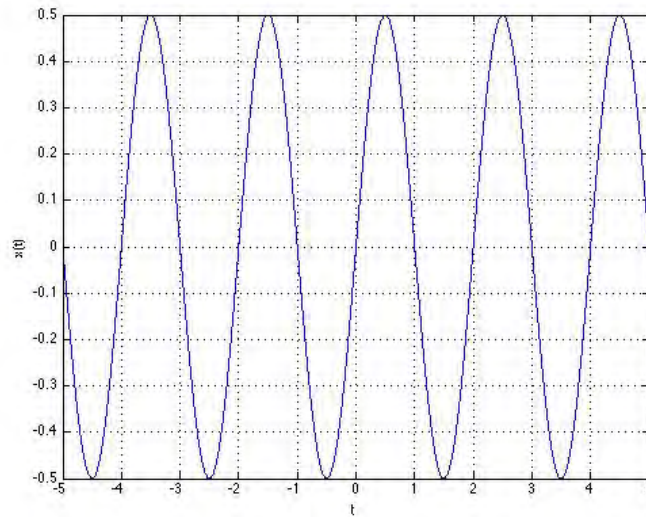$x_o(t) = \frac{1}{2}sin(\pi t)u(t) + \frac{1}{2}sin(\pi t)(1 - u(t))$

$x_o(t) = \frac{1}{2}sin(\pi t)$

We see that the odd portion of this signal is of sinusoidal form with half the amplitude of input signal (for time $t > 0$).

∴ the odd portion of x(t) is **periodic**.

We can verify this in MatLab by plotting the following:

```
t = -5:0.001:5
x = 1/2*[sin(pi.*t).*(t>=0) - sin(-pi.*t).*(t<0)]
plot(t, x)
xlabel('t'); ylabel('x(t)'); % labels
grid;
```
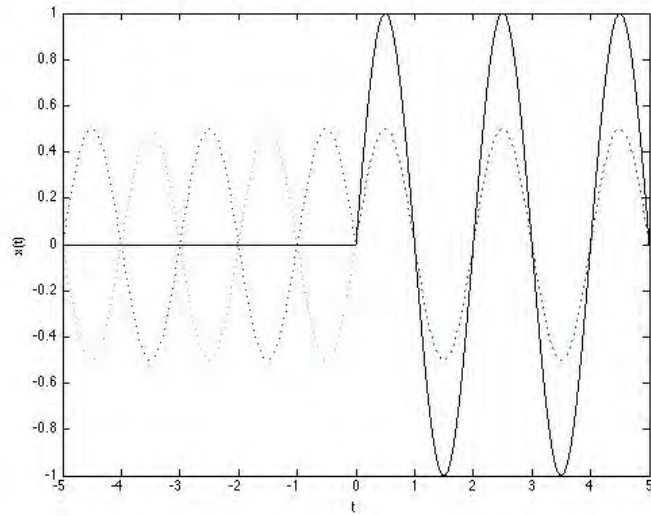

Furthermore, as a simple exercise, we can show that the sum of even and odd signals will sum to the original signal.

```
t = -5:0.001:5
xo = 1/2*[sin(pi.*t).*(t>=0) - sin(-pi.*t).*(t<0)]
xe = 1/2*[sin(pi.*t).*(t>=0) + sin(-pi.*t).*(t<0)]
plot(t, xo, 'g:')
hold on;
plot(t,xe, 'b:')
plot(t, xo + xe, 'black')
xlabel('t'); ylabel('x(t)'); % labels
grid;
```

Question 2

a)

$$\text{Backward differencer: } y[t] = x[t] - x[t-1] \qquad (1)$$

Linearity:

To test linearity we check whether superposition holds. That is,

$ax_1[t] + bx_2[t] \rightarrow ay_1[t] + by_2[t]$

let $x[t] = ax_1[t] + bx_2[t]$ and sub into (1),

$= (ax_1[t] + bx_2[t]) - (ax_1[t-1] + bx_2[t-1])$

$= a(x_1[t] - x_1[t-1]) + b(x_2[t] - x_2[t-1])$

$= ay_1[t] + by_2[t]$

$\therefore$ The system is **linear**.

Time Invariance:

To test this system for time-invariance, we first delay the input by δ

$x_d[t] = x[t+\delta]$

$y_1[t] = x_d[t] - x_d[t-1] = x[t+\delta] - x[t-1+\delta]$

Now we delay the output by δ

$y_2[t] = y[t+\delta] = x[t+\delta] - x[t-1+\delta]$

We now see that $y_1[t] == y_2[t]$.

$\therefore$ The system is **time-invariant**.

Causality:

To test causality, we check that the output at any time t_0 depends only on the value of the input x(t) for $t <= t_0$.

We see that the output of $y[t_0]$ depends only on input values to x(t) at times t_0 and $(t_0 - 1)$ (meeting the above condition).

$\therefore$ The system is **causal**.

b)

$$\text{Forward differencer: } y[t] = x[t+1] - x[t] \qquad (2)$$

Linearity:

To test linearity we check whether superposition holds. That is,

$ax_1[t] + bx_2[t] \rightarrow ay_1[t] + by_2[t]$

let $x[t] = ax_1[t] + bx_2[t]$ and sub into (2),

$= (ax_1[t+1] + bx_2[t+1]) - (ax_1[t] + bx_2[t])$

$= a(x_1[t+1] - x_1[t]) + b(x_2[t+1] - x_2[t])$

$= ay_1[t] + by_2[t]$

$\therefore$ the system is **linear**.

Time Invariance:

To test this system for time-invariance, we first delay the input by δ

$x_d[t] = x[t+\delta]$

$y_1[t] = x_d[t+1] - x_d[t] = x[t+\delta+1] - x[t+\delta]$

Now we delay the output by δ

$y_2[t] = y[t+\delta] = x[t+\delta+1] - x[t+\delta]$

We now see that $y_1[t] == y_2[t]$.

∴ The system is **time-invariant**.

Causality:

To test causality, we check that the output at any time t_0 depends only on the value of the input x(t) for $t <= t_0$.

We see that the output of $y[t_0]$ depends only on input values to x(t) at times $(t_0 + 1)$ and t_0. Therefore the system's output depends on an input value from future time.

∴ The system is **non-causal**.

c)

$$\text{Central differencer: } y[t] = x[t + \tfrac{1}{2}] - x[t - \tfrac{1}{2}] \qquad (3)$$

Linearity:

To test linearity we check whether superposition holds. That is,

$ax_1[t] + bx_2[t] \rightarrow ay_1[t] + by_2[t]$

let $x[t] = ax_1[t] + bx_2[t]$ and sub into (3),

$= (ax_1[t + \tfrac{1}{2}] + bx_2[t + \tfrac{1}{2}]) - (ax_1[t - \tfrac{1}{2}] + bx_2[t - \tfrac{1}{2}])$

$= a(x_1[t + \tfrac{1}{2}] - x_1[t - \tfrac{1}{2}]) + b(x_2[t + \tfrac{1}{2}] - x_2[t - \tfrac{1}{2}]$

$= ay_1[t] + by_2[t]$

∴ the system is **linear**.

Time Invariance:

To test this system for time-invariance, we first delay the input by δ

$x_d[t] = x[t + \delta]$

$y_1[t] = x_d[t + \tfrac{1}{2}] - x_d[t - \tfrac{1}{2}] = x[t + \delta + \tfrac{1}{2}] - x[t + \delta - \tfrac{1}{2}]$

Now we delay the output by δ

$y_2[t] = y[t + \delta] = x[t + \delta + \tfrac{1}{2}] - x[t + \delta - \tfrac{1}{2}]$

We now see that $y_1[t] == y_2[t]$.

∴ The system is **time-invariant**.

Causality:

To test causality, we check that the output at any time t_0 depends only on the value of the input x(t) for $t <= t_0$.

We see that the output of $y[t_0]$ depends only on input values to x(t) at times $(t_0 + \tfrac{1}{2})$ and $(t_0 - \tfrac{1}{2})$. Therefore the system's output depends on an input value from future time.

∴ The system is **non-causal**.

Question 3

The energy stored in a capacitor can be expressed as

$W = q\Delta V_c \rightarrow dW = V_c dq \rightarrow W = \int_0^Q V_c dq \rightarrow W = \int_0^Q \frac{q}{C} dq$

$W = \frac{Q^2}{2C} = \frac{1}{2C}(CV_c)^2 = \frac{1}{2}CV_c^2$

(http://hyperphysics.phy-astr.gsu.edu/hbase/electric/capeng.html)

The voltage across the capacitor when it is fully charged will be equal to that of V_s. Therefore, rearranging this equation and subbing in $W = 150J, C = 68uF$ we get:

$V_s = \sqrt{\frac{2W}{C}} = \sqrt{\frac{2*150J}{68uF}} = 2100.42V$

This is the voltage required across the given capacitor to store 150J.

When the switch is thrown to "D", the capacitor is discharged through $R_{chest} = 10k\Omega$. The final voltage value on the capacitor after 95J has been delivered is

$V_s = \sqrt{\frac{2*(150-95)J}{68uF}} = 1271.868V$

Before proceeding, we can check that fibrillation will still occur at this voltage, with $R_{chest} = 10k\Omega$, the current experienced across the chest at the end of the discharge will still be $i_c = \frac{1271.868}{10000} = 127.1mA$. This is greater than the minimum fibrillation current necessary (75ma - as per question 5).

Now we can relate capacitor voltage to current:

$C = \frac{Q}{V_c} = \frac{\frac{dQ}{dt}}{\frac{dV_c}{dt}} = \frac{I_c}{\frac{dV_c}{dt}} \rightarrow \frac{dV_c}{dt} = \frac{1}{C}I_c \rightarrow I_c = C\frac{dV_c}{dt}$

We note that the KVL equation for the RC circuit is:

$V_c(t) + I_c(t)R = 0$

$V_c(t) + CR\frac{dV_c}{dt} = 0$

$\int \frac{1}{V_C} dV_c = \int -\frac{1}{CR} dt$

$ln(V_c) = -\frac{1}{CR}t + D$; where D is unknown

$V_c = e^{-\frac{1}{CR}t+D}$; Using $V_0 = e^{-D}$ we get,

$V_c = V_0 e^{-\frac{1}{CR}t}$

We know our the voltage initially required to hold a 150J dose. We also know the remaining voltage after 95J has been delivered to the victim. Therefore we can solve for time t.

$1271.868 = 2100.42e^{-\frac{t}{10K*68uF}}$

$ln\frac{1271.868}{2100.42} = -\frac{t}{0.68}$

$t = 0.3411 seconds$

As shown above, it will take 0.3411 seconds to deliver a 95J dose.

Question 4

a) Strictly speaking, the Nyquist frequency is the half the sampling frequency of a discrete signal (http://en.wikipedia.org/wiki/Nyquist_frequency). On the other hand, the Nyquist rate is two times the bandwidth of a band limited signal. The Nyquist rate and the Nyquist frequency should not be confused. We know that the bandwidth of the signal is 3Mhz (3.5-0.5Mhz). Therefore, the Nyquist rate is equal to 6Mhz, corresponding the minimum sampling rate required to avoid aliasing. However, using regular sampling techniques, the frequency which Nyquist states a signal should be sampled at is twice that of the highest frequency contained within the signal. Therefore, in a frequency swept signal with the highest frequency at 3.5Mhz, the frequency of sampling under Nyquist is equal to 7Mhz.

b) A noise signal of 2.4Ghz (significantly greater than the measured frequency) will still effect the samples unless the signal is passed through a low pass filter with a cutoff frequency selected such that the 2.4Ghz signal is highly attenuated, whilst the low frequency signal is still passed through unaffected. Without filtering, aliasing of the high frequency wave will occur.

For the interested reader:
The Touche system does not actually sample this signal at this speed, as having an ADC this quick would cost a small fortune. The device measures the returned signal in terms of average voltage in differing frequency bands. The signal is fed to an isolation buffer before entering an envelope detector which converts the AC signal into a time varying DC signal which is then sampled by a 200Khz ADC. Therefore, in a sense, they did not require to sample at greater than 7Mhz, as they did not care about signal reconstruction, but rather used analog circuitry to find the average amplitude associated with the different frequencies.
For a full report on the quite amazing system, you can find the paper here:
http://www.kevinli.net/courses/mobilehci_w2013/papers/touche.pdf

Question 5

Before beginning this question, several assumptions must be clarified. Firstly, it is unclear whether Uncle Robert measured resistance through a single shoe or two shoes. For the sake of argument we will assume that he had some intellect and found the equivelent resistance of both shoes. (Either way the answer will be fundamentally the same). When modelled as a pure resistor with the current path going through his shoes, Uncle Robert's argument holds with only $\frac{240}{1M\Omega} = 0.24\text{mA}$ flowing through his body.

It should be noted, if Uncle Robert has a fast switching safety switch, defibrillation may not occur anyway.

It becomes apparent that Uncle Robert's simple highly resistive model is not complete.

Firstly, the large surface area of his body will have capacitive interactions with the ground and/or wall. In this light, the human body itself has some capacitive properties (evidenced by electrostatic precautions required when working with sensitive components). In fact, the Electrostatic Discharge Association (EDA) models the human body as a 100pF capacitor in series with a 1.5k resistor (http://en.wikipedia.org/wiki/ Body_capacitance). Other sources note that high voltage electrical energy has been found to quickly break down human skin, reducing the body's resistance dramatically (Reference: http://en.wikipedia.org/wiki/Electric_shock #Body_resistance). In fact, a study has found that at 220V (close to this example), 95% of a surveyed population has a resistance of approximately 2125Ω, which is undeniably close to the 1.5k stated previously (Reference: Reilly, J. Patrick (1998). Applied Bioelectricity: From Electrical Stimulation to Electropathology (2nd ed.)).

If we do follow the EDA's model, we would find that the equivalent impedance of the body is:

$$Z_{body} = 1500 + \frac{1}{j2\pi50*100*10^{-9}} = (1500 - 31831J)\Omega$$

And the total current flowing in his body would be:

$$I_{body} = \frac{240}{10^6+1500-31831J} = 2.3952*10^{-4}\angle1.82A$$

Which is not much less that the original model (Defibrillation will not occur). Uncle Robert has also not taken into account the multiple scenarios of electrocution. This is important as current can flow through his body in two ways; from arm to arm or arm to ground. This is because there is a neutral conductor also going to the lamp inside the wall. Therefore, under conditions of arm-arm contact, defibrillation will occur ($\frac{240}{approx.\ 2k} = 120mA$).

Question 6

a)

x(t) is our continuous signal to be sampled. We want to find x[n] that is periodic. For x[n]=x(nT) to be periodic, it must hold that:

$$x[n] = x[n + n_0] \rightarrow x(nT) = x((n + n_0)T) \qquad (1)$$

where n_0 is an integer, as x[n] only takes integer arguments.

We can find both the L.H.S. and the R.H.S. of the above equation knowing that $x(t) = e^{jw_0 t}$.

$$x[n + n_0] = x((n + n_0)T) = e^{j\omega_0(n+n_0)T} = e^{j\omega_0 nT} * e^{j\omega_0 n_0 T} \qquad (2)$$

$$x[n] = x(nT) = e^{j\omega_0 nT} \qquad (3)$$

But in order for (1) to hold, (2) must equal (3):

$$e^{j\omega_0 nT} * e^{j\omega_0 n_0 T} = e^{j\omega_0 nT}$$

Which is only true if:

$$e^{j\omega_0 n_0 T} = 1 = e^{j2\pi k}, \quad k \in \mathbb{I}$$

$$\therefore w_0 n_0 T = 2\pi k \text{ with } w_0 = \frac{2\pi}{T_0}$$

$$\frac{2\pi}{T_0} n_0 T = 2\pi k$$

$$\rightarrow \frac{T}{T_0} = \frac{k}{n_0}$$

With both k and n_0 being integers, we can see that $\frac{T}{T_0}$ must be a rational number.

b)

The fundamental period is the smallest value of n_0 for which the following equation holds:

$$x[n] = x[n + n_0]$$

We know from part (a) that in order for this to hold, $\frac{T}{T_0} = \frac{k}{n_0} = \frac{p}{q}$

We can therefore stipulate that the fundamental period of x[n] is in the smallest n_0 such that $n_0 \frac{T}{T_0} = n_0 \frac{p}{q} = k$

The smallest n_0 such that $n_0 p$ has a divisor q is also known as the least common multiple of p and q, divided by p, Thus,

$$n_0 = \frac{LCM(p,q)}{p}$$

Since $k = \frac{p}{q} n_0$, we see that $k = \frac{LCM(p,q)}{q}$

The fundamental frequency can be simply stated as

$$\Omega_0 = \frac{2\pi}{n_0} \text{ where } T_0 = \frac{2\pi}{\omega_0}$$

Therefore, using the same argument as above, we find:

$$\Omega_0 = \frac{2\pi T}{k T_0} = \frac{1}{k} w_0 T = \frac{q}{LCM(p,q)} w_0 T$$

c)

In order to find the number of periods of x(t) needed to obtain the samples that form a single period, we need to find the smallest integer value 'm' such that

$$x[n + n_0] = x(nT + mT_0) \text{ and } x[n + n_0] = x((n + n_0)T)$$

Therefore, $n_0 = m\frac{T_0}{T}$

We note that n_0 must be an integer, and therefore $m\frac{T_0}{T}$ must be an integer also. Furthermore, $m\frac{q}{p}$ must then also be an integer as per part (b).

Thus $m = \frac{LCM(p,q)}{q}$.

d)

When arbitrarily spacing the samples of x(t) or implementing asynchronous sampling, we can reduce the number of samples needed for signal reconstruction. One method of doing this is using a time encoder that encodes the times when preset signal changes occur, as opposed to constant nyquist rate sampling. Therefore, asynchronous sampling schemes can be advantageous as it can reduce the samples required to reconstruct our data. Of course this may require us to take into consideration the nature of the signals when designing the ADC's for a specific application. Below is a good visualisation of Asynchronous sampling.
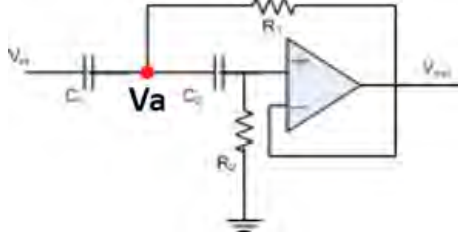

http://www.cinc.org/archives/2012/pdf/0061.pdf
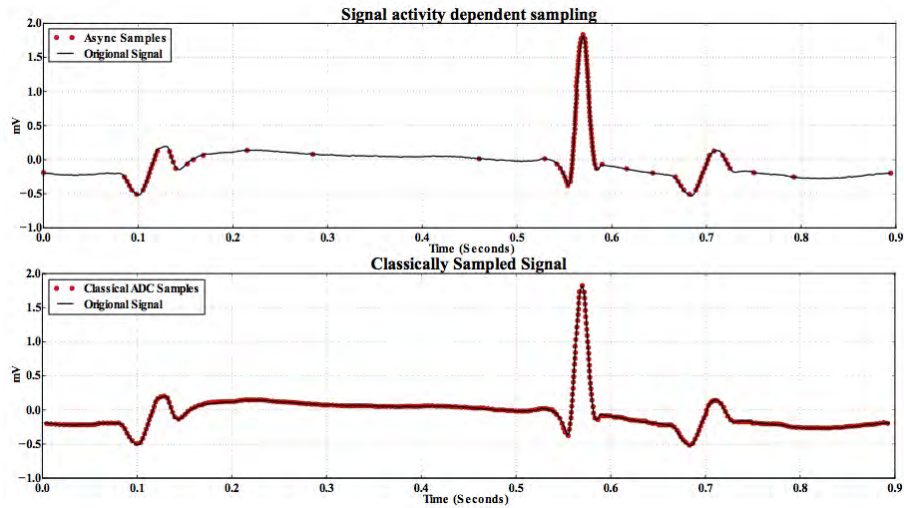
A disadvantage of asynchronous sampling is that it is more complex to implement. You should not run to the patent office quite yet. There are plenty of patents out on asynchronous samplers already, e.g. http://www.google.com/patents/US7327302

Question 7

a)

We will label V_a as shown in the following diagram:

We observe that the op-amp is connected with negative feedback. Therefore we can make the assumption that $V_- = V_+$. We also see that $V_- = V_{out}$.

We perform KCL analysis at node A:

$\frac{V_a - V_{in}}{\frac{1}{sC_1}} + \frac{V_a - V_+}{\frac{1}{sC_2}} + \frac{V_a - V_{out}}{R_1} = 0$

$V_a(sC_1 + sC_2 + \frac{1}{R_1}) = V_{in}sC_1 + V_{out}(sC_2 + \frac{1}{R_1})$ (1)

We perform KCL analysis at node V_+

$\frac{V_+ - V_a}{\frac{1}{sC_2}} + \frac{V_+}{R_2} = 0$

$V_+(sC_2 + \frac{1}{R_2}) = V_a sC_2$

$V_a = V_{out}\frac{sC_2 + \frac{1}{R_2}}{sC_2}$

Subbing this result into equation (1), we solve for the transfer function $H(s)$

$V_{out}\frac{sC_2 + \frac{1}{R_2}}{sC_2}(sC_1 + sC_2 + \frac{1}{R_1}) = V_{in}sC_1 + V_{out}(sC_2 + \frac{1}{R_1})$

$V_{out}[\frac{sC_2 + \frac{1}{R_2}}{sC_2}(sC_1 + sC_2 + \frac{1}{R_1}) - (sC_2 + \frac{1}{R_1})] = V_{in}sC_1$

$\frac{V_{out}}{V_{in}} = \frac{sC_1}{\frac{sC_2 + \frac{1}{R_2}}{sC_2}(sC_1 + sC_2 + \frac{1}{R_1}) - (sC_2 + \frac{1}{R_1})} = \frac{s^2 C_1 C_2}{(sC_2 + \frac{1}{R_2})(sC_1 + sC_2 + \frac{1}{R_1}) - sC_2(sC_2 + \frac{1}{R_1})}$

$= \frac{s^2 C_1 C_2}{(s^2 C_2 C_1 + s^2 C_2^2 + sC_2\frac{1}{R_1} + \frac{1}{R_2}sC_1 + \frac{1}{R_2}sC_2 + \frac{1}{R_2 R_1} - s^2 C_2^2 - \frac{sC_2}{R_1})}$

$\frac{V_{out}}{V_{in}} = \frac{s^2 C_1 C_2}{s^2 C_2 C_1 + \frac{1}{R_2}sC_1 + \frac{1}{R_2}sC_2 + \frac{1}{R_2 R_1}} = \frac{s^2}{s^2 + \frac{1}{R_2}(\frac{1}{C_2} + \frac{1}{C_1})s + \frac{1}{R_2 R_1 C_1 C_2}}$

$\rightarrow$ This looks like a High Pass Filter, as there are two zeros at 0, and 2 poles in the negative half of the plane.

b)

When determining the stability of the circuit, we will firsty find stability of the general circuit (without values used).

We look at the characteristic equation of the high pass sallen key filter:

$s^2 + \frac{1}{R_2}(\frac{1}{C_2} + \frac{1}{C_1})s + \frac{1}{R_2 R_1 C_1 C_2}$

In order to determine the stability of the system, we will apply the Routh-Hurwitz method (as introduced in METR3200). Firstly we take the characteristic equation of the system as stated above and formulate the Routh Hurwitz table/array as follows:

$s^2\|$	1	$\frac{1}{R_1 R_2 C_1 C_2}$	0
$s^1\|$	$\frac{1}{R_2}(\frac{1}{C_2} + \frac{1}{C_1})$	0	0
$s^0\|$	$\frac{1}{R_1 R_2 C_1 C_2}$	0	0

The stability criteria outlines that the number of sign changes in the first column corresponds to the number of non-negative poles. We see that the first entry in the first column is 1, and therefore, for stability to exist (no non-negative poles), the following must be true:

$\frac{1}{R_2}\left(\frac{1}{C_2} + \frac{1}{C_1}\right) > 0, \; \frac{1}{R_1 R_2 C_1 C_2} > 0$

In order to find the stability of our specific case of $C_1 = 0.01uF$ and $R_1 = 10k\Omega$, we find that:

$\frac{1}{R_2 C_2} + \frac{10^8}{R_2} > 0, \; \frac{10^4}{R_2 C_2} > 0$

We immediately see that for $R_2 > 0, C_2 > 0$ these conditions will always hold and therefore for all practical passive resistor and capacitor components, the circuit is stable.

$\therefore$ the system is stable for all values.

c)

We firstly look at the characteristic equation of the closed loop system equation using $C_1 = 0.01uF$ and $R_1 = 10k\Omega$.

$s^2 + \frac{1}{R_2 C_2}s + \frac{10^8}{R_2}s + \frac{10^4}{R_2 C_2} = 0$

There are 2 system parameters which can be varied to find the root loci of this system. This will lead to infinite root locus plots. We will therefore offer a more analytical approach and show the root locus on a case by case basis. Firstly, we pick our varying system gain parameter as:

$\alpha = \frac{1}{R_2 C_2}$

This gives us our root locus equation:

$s^2 + \alpha s + \frac{10^8}{R_2}s + \alpha 10^4 = 0$

Rearranging this into Root Locus form, we get:

$\alpha(s + 10^4) = -(s^2 + \frac{10^8}{R_2}s)$

$\alpha \frac{(s+10^4)}{s(s+\frac{10^8}{R_2})} = -1$

We note the open loop poles and zeros of the equation:

Zeros: -10^4

Poles: $0, \frac{-10^8}{R_2}$

We see immediately that there are two cases which will change the root locus plot arrangement. Namely, it is when the pole dependant on R_2 is greater than or less than the zero on the the left hand side plane.

We must emphasise that by varying the system parameter alpha, and also varying R_2, we have a complete root locus plot for both R_2 and C_2. However, through utilising the two different cases outlined above, we can draw two root locus plots that generalise the system.

We can draw the root locus for cases the following MatLab Code:

```
r2 = 15000
num = [1 10^4];
den = [1 10^8/r2 0];
sys2 = tf(num, den)
subplot(2,1,1)
rlocusplot(sys2); % Draw root locus

r2 = 5000
num = [1 10^4];
den = [1 10^8/r2 0];
sys2 = tf(num, den)
subplot(2,1,2)
rlocusplot(sys2); % Draw root locus
```
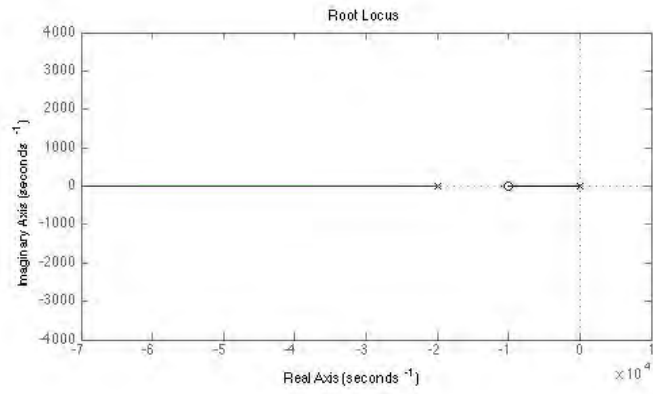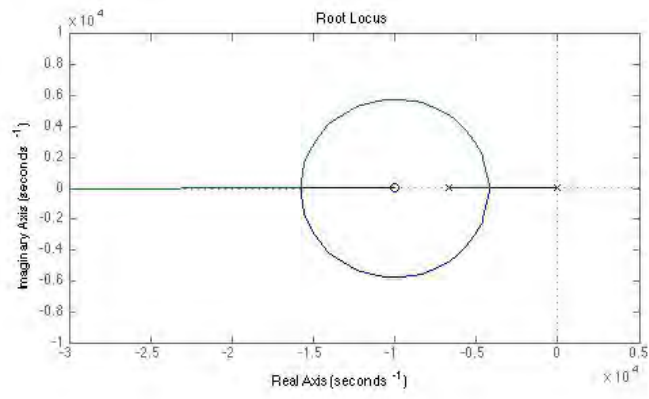
The top plot corresponds to picking $R_2 = 15000$ and the bottom plot corresponds to picking $R_2 = 5000$, a clear dividing case.

Question 8

Before discussing this question, we first find the solution to the first order differential and find N (the number of atoms) in terms of t (time).

$\frac{dN}{dt} = -\lambda N$

$\int_{N(0)}^{N} \frac{dN}{N} = -\int_0^t \lambda dt$

$ln(n)|_{N(0)}^{N} = -\lambda(t - 0)$

$ln(\frac{N}{N(0)}) = -\lambda t$

$\frac{N}{N(0)} = -\lambda t$

$\therefore N(t) = N(0)e^{-\lambda t}$

a)

In order to find the percentage left after one lifetime (τ) decay, we find $N(\tau)$.

$\tau = \frac{1}{\lambda}$

$\therefore N(\tau) = N(0)e^{-\lambda \tau} = N(0)e^{-1}$

Therefore the fraction atoms left after one lifetime is:

$\frac{N}{N(0)} = e^{-1} = 0.36788$ or 36.788%

b)

Firstly, we recall that $t_{\frac{1}{2}} = \tau ln(2) = \frac{ln(2)}{\lambda}$

In order to find concentrations to reach 10%, we do the following:

$\frac{N}{N(0)} = 10\% = e^{-\lambda t}$

$ln(0.1) = -2.30259 = -\lambda t$

$t = \frac{ln(10)}{\lambda}$ with $\lambda = \frac{ln(2)}{t_{\frac{1}{2}}}$

$\therefore t = \frac{ln(10)}{ln(2)} t_{\frac{1}{2}} = 3.32193 t_{\frac{1}{2}}$

It will take approximately 3.322 half lifes for decay to 10%.

(c) and (d)

Firstly, we will label the lead-211 decay process as 1. The Bismuth-211 decay as process 2. The Thallium-207 decay as process 3. We note that Lead-207 is the final state, as it is stable.

Now we can write the system of differential equations as:

$\frac{dN_1}{dt} = -\lambda_1 N_1$

$\frac{dN_2}{dt} = -\lambda_2 N_2 + \lambda_1 N_1$

$\frac{dN_3}{dt} = -\lambda_3 N_3 + \lambda_2 N_2$

$\frac{dN_4}{dt} = \lambda_3 N_3$

Further, from the decay half-lifes we know that: $\lambda = \frac{ln(2)}{t_{1/2}}$

$\lambda_1 = \frac{ln(2)}{36} = 0.019254088$

$\lambda_2 = \frac{ln(2)}{2.1} = 0.330070086$

$\lambda_3 = \frac{ln(2)}{4.8} = 0.144405662$

We know that initially the atom count of Bismuth-211. Thallium-207, and Lead-207 are 0. Furthermore, we also know that Lead is starting at 1kg (our normalised quantity).

$N_1(0) = 1$

$N_2(0) = 0$

$N_3(0) = 0$

$N_4(0) = 0$

In order to find the solution to the system of equations above, we turn to our good friend Pierre-Simon Laplace. Using laplace transforms we can easily manipulate this system algebraically.

$sN_1 - N_1(0) = -\lambda_1 N_1$

$sN_2 - N_2(0) = -\lambda_2 N_2 + \lambda_1 N_1$

$sN_3 - N_3(0) = -\lambda_3 N_3 + \lambda_2 N_2$

$sN_4 - N_4(0) = \lambda_3 N_3$

Taking into account null initial conditions, these equations simplify to:

$sN_1 - N_1(0) = -\lambda_1 N_1$

$sN_2 = -\lambda_2 N_2 + \lambda_1 N_1$

$sN_3 = -\lambda_3 N_3 + \lambda_2 N_2$

$sN_4 = -\lambda_4 N_4 + \lambda_3 N_3$

Note: Although these equations can be trivially solved using Bateman's equations, we will do a quick derivation for the reader. We find the solution firstly for N_1:

$N_1(s) = \frac{N_1(0)}{s+\lambda_1}$

$\rightarrow N_1(t) = N_1(0)e^{-\lambda_1 t}$

Using this, we find the solution for N_2:

$N_2(s) = \frac{\lambda_1 N_1}{s+\lambda_2} = \frac{\lambda_1 N_1(0)}{(s+\lambda_2)(s+\lambda_1)} = \frac{\lambda_1 N_1(0)}{\lambda_2-\lambda_1}\frac{1}{s+\lambda_1} + \frac{\lambda_1 N_1(0)}{\lambda_1-\lambda_2}\frac{1}{s+\lambda_2}$

$N_2(s) = N_1(0)\frac{\lambda_1}{\lambda_1-\lambda_2}(\frac{1}{s+\lambda_2} - \frac{1}{s+\lambda_1})$

$\rightarrow N_2(t) = N_1(0)\frac{\lambda_1}{\lambda_1-\lambda_2}(e^{-\lambda_2 t} - e^{-\lambda_1 t})$

Using this, we find the solution for N_3:

$N_3(s) = \frac{\lambda_2 N_2}{s+\lambda_3} = \frac{\lambda_1\lambda_2 N_1(0)}{(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)} = \lambda_1\lambda_2 N_1(0)[\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\frac{1}{s+\lambda_1} + \frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\frac{1}{s+\lambda_2} + \frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\frac{1}{s+\lambda_3}]$

$\rightarrow N_3(t) = \lambda_1\lambda_2 N_1(0)[\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}e^{-\lambda_1 t} + \frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}e^{-\lambda_2 t} + \frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}e^{-\lambda_3 t}]$

Using this, we find the solution for N_4:

$N_4(s) = \frac{\lambda_3 N_3}{s} = \frac{\lambda_1\lambda_2\lambda_3 N_1(0)}{s(s+\lambda_1)(s+\lambda_2)(s+\lambda_3)} = \lambda_1\lambda_2\lambda_3 N_1(0)[\frac{1}{\lambda_1\lambda_2\lambda_3}\frac{1}{s} + \frac{1}{-\lambda_1(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\frac{1}{s+\lambda_1} + \frac{1}{-\lambda_2(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\frac{1}{s+\lambda_2} + \frac{1}{-\lambda_3(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\frac{1}{s+\lambda_3}]$

$N_4(t) = \lambda_1\lambda_2\lambda_3 N_1(0)[\frac{1}{\lambda_1\lambda_2\lambda_3}u(t) + \frac{1}{-\lambda_1(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}e^{-\lambda_1 t} + \frac{1}{-\lambda_2(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}e^{-\lambda_2 t} + \frac{1}{-\lambda_3(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}e^{-\lambda_3 t}]$

$\rightarrow N_4(t) = N_1(0) - \frac{\lambda_2\lambda_3 N_1(0)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}e^{-\lambda_1 t} - \frac{\lambda_1\lambda_3 N_1(0)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}e^{-\lambda_2 t} - \frac{\lambda_1\lambda_2 N_1(0)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}e^{-\lambda_3 t}$

for $t > 0$

Note how we needed to work through the representations of N_1, N_2 and N_3 in order to find N_4.

Now we have time domain equations that represent the concentrations of each isotope. We can now plot these using MatLab:

```matlab
t = 0:0.01:360
n1 = 1 % normalised quantity initial condition

halflife1 = 36 %minutes
halflife2 = 2.1 %minutes
halflife3 = 4.8 %minutes

lambda1 = log(2)/halflife1
lambda2 = log(2)/halflife2
lambda3 = log(2)/halflife3


N1 = n1.*exp(-lambda1.*t)
N2 = n1*lambda1/(lambda1-lambda2)*(exp(-lambda2*t) - exp(-lambda1*t))
N3 = n1*lambda1*lambda2*[exp(-lambda1*t)/((lambda2-lambda1)*(lambda3-lambda1)) ...
    + exp(-lambda2*t)/((lambda1-lambda2)*(lambda3-lambda2)) ...
    + exp(-lambda3*t)/((lambda1-lambda3)*(lambda2-lambda3))]
N4 = n1 - lambda2*lambda3*n1*exp(-lambda1*t)/((lambda2-lambda1)*(lambda3-lambda1)) ...
    - lambda1*lambda3*n1*exp(-lambda2*t)/((lambda1-lambda2)*(lambda3-lambda2)) ...
    - lambda1*lambda2*n1*exp(-lambda3*t)/((lambda1-lambda3)*(lambda2-lambda3))

hold on;
plot(t, N1, 'g')
plot(t, N2, 'b')
plot(t, N3, 'r')
plot(t, N4, 'black')
ylabel('Normalised Concentration'); xlabel('Time in Minutes'); grid;
```
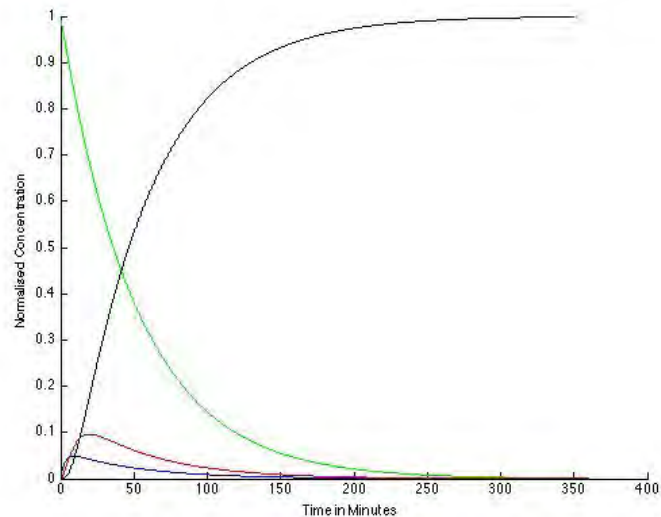

Here we can see the concentration levels of Lead-211 (green), Bismuth-211 (blue), Thallium-207 (red), Lead-207 (Black).

Now we solve for t to find how long it takes for 1Kg of Lead-211 to turn into 0.9Kg of Lead-207.

We use the MatLab 'solve' tool to do this calculation as follows:

```matlab
n1 = 1; % normalised quantity initial condition

halflife1 = 36; %minutes
halflife2 = 2.1; %minutes
halflife3 = 4.8; %minutes

lambda1 = log(2)/halflife1;
lambda2 = log(2)/halflife2;
lambda3 = log(2)/halflife3;

syms t
solve(0.1 - lambda2*lambda3*n1*exp(-lambda1*t)/((lambda2-lambda1)*(lambda3-lambda1)) ...
    - lambda1*lambda3*n1*exp(-lambda2*t)/((lambda1-lambda2)*(lambda3-lambda2)) ...
    - lambda1*lambda2*n1*exp(-lambda3*t)/((lambda1-lambda3)*(lambda2-lambda3)), t)


ans =

130.14326082123215849137700625677
```

Therefore, it will take approximately 130.143 minutes for 1000 grams of Lead-211 to turn into 900 grams of Lead-207. Equivalently, this is approximately 7808.6 seconds.

e)

The radiation emission of a single of Lead-211 to Bismuth-211 nuclide will be proportional to the percentage of lead-211 multiplied by its decay rate. Therefore the relationship of total radiation can be expressed as:

$\gamma = N_1 \lambda_1 + N_2 \lambda_2 + N_3 \lambda_3$

$\gamma = N_1(0)[\frac{1}{s+\lambda_1}\lambda_1 + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}(\frac{1}{s+\lambda_2} - \frac{1}{s+\lambda_1}) + \lambda_1\lambda_2\lambda_3[\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\frac{1}{s+\lambda_1} + \frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\frac{1}{s+\lambda_2} + \frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\frac{1}{s+\lambda_3}]]$

$\gamma = N_1(0)[\lambda_1 e^{-\lambda_1 t} + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}(e^{-\lambda_2 t} - e^{-\lambda_1 t}) + \lambda_1\lambda_2\lambda_3[\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}e^{-\lambda_1 t} + \frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}e^{-\lambda_2 t} + \frac{1}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}e^{-\lambda_3 t}]]$

$\gamma = e^{-\lambda_1 t}[\lambda_1 + \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} + \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}] + e^{-\lambda_2 t}(\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} + \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}) + e^{-\lambda_3 t}\frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$

$\gamma = 0.0633 * e^{-0.0193t} - 0.0045 * e^{-0.3301t} - 0.0395 * e^{-0.1444t}$

Again, using matlab we can plot the graph of this as:

```
t = 0:0.01:360;
nl = 1; % normalised quantity initial condition

halflife1 = 36; %minutes
halflife2 = 2.1; %minutes
halflife3 = 4.8; %minutes

lambda1 = log(2)/halflife1
lambda2 = log(2)/halflife2
lambda3 = log(2)/halflife3

P = [lambda1 + lambda1*lambda2/(lambda2 - lambda1) ...
    + lambda1*lambda2*lambda3/((lambda2-lambda1)*(lambda3 - lambda1))]...
    * exp(-lambda1*t) + [lambda1*lambda2/(lambda1 - lambda2) ...
    + lambda1*lambda2*lambda3/((lambda1-lambda2)*(lambda3 - lambda2))]...
    * exp(-lambda2*t) + [lambda1*lambda2*lambda3/((lambda1-lambda3)...
    *(lambda2 - lambda3))]*exp(-lambda3*t);

plot(t, P)
ylabel('Radiation'); xlabel('Time in Minutes'); grid;
```