A Discussion of Sampling Theorems*

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Summary—**The convolution theorem of Fourier analysis is a convenient tool for the derivation of a number of sampling theorems. This approach has been used by several authors to discuss first-order sampling of functions whose spectrum is limited to a region including the origin ("low-pass" functions). The present paper extends this technique to several other cases: second-order sampling of low-pass and band-pass functions, quadrature and Hilbert-transform sampling, sampling of periodic functions, and simultaneous sampling of a function and of one or more of its derivatives.**

Introduction

Several sampling theorems have appeared in the engineering literature.[1–5] These may be derived in a particularly perspicuous manner by means of the convolution theorem of Fourier analysis. The sampling process is regarded as a multiplication by a periodic sequence of δ-functions, its counterpart in the frequency domain being a convolution by a train of equispaced δ-functions. Interpolation—the recovery of the original signal from its sample values—is viewed in the frequency domain as a process of reconstructing the original spectrum by means of a spectral "window." The corresponding time domain operation consists of the convolution of the sample impulses with the inverse Fourier transform of the window function. This approach has been used by a number of authors[6–8] to discuss the equispaced sampling of low-pass functions. It is the purpose of this paper to present a consistent set of heuristic derivations for a number of additional sampling theorems.

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[1] C. E. Shannon, "Communication in the presence of noise," Proc. IRE, vol. 37, pp. 10–21; January, 1949.

[2] A. Kohlenberg, "Exact interpolation of band-limited functions," *J. Appl. Phys.*, vol. 24, pp. 1432–1436; December, 1953.

[3] S. Goldman, "Information Theory," Prentice-Hall, Inc., New York, N. Y.; 1953.

[4] L. J. Fogel, "A note on the sampling theorem," IRE Trans. on Information Theory, vol. IT-1, pp. 47–48; March, 1955.

[5] D. L. Jagerman and L. J. Fogel, "Some general aspects of the sampling theorem," IRE Trans. on Information Theory, vol. IT-2, pp. 139–146; December, 1956.

[6] P. M. Woodward, "Probability and Information Theory, with Applications to Radar," McGraw-Hill Book Co., Inc., New York, N. Y.; 1955.

[7] R. B. Blackman and J. W. Tukey, "The measurement of power spectra from the point of view of communication engineering," *Bell Sys. Tech. J.*, vol. 37, pp. 185–280, 485–569; January and March, 1958.

[8] J. R. Ragazzini and G. F. Franklin, "Sampled Data Control Systems," McGraw-Hill Book Co., Inc., New York, N. Y.; 1958.

The following transform definitions will be used:

$$F(f) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt, \qquad \omega \equiv 2\pi f$$

$$f(t) = \int_{-\infty}^{+\infty} F(f)e^{i\omega t}df.$$

It will be convenient to use the notation

$$a(t) * b(t) \equiv \int_{-\infty}^{+\infty} a(\tau)b(t-\tau)d\tau.$$

Following the nomenclature of Kohlenberg,[2] sampling of a time function[9] will be designated as first-order if the sample points are equispaced. Second-order sampling involves two interleaved sequences of equispaced sampling points.

Sampling of Low-Pass Functions

The simplest case is that of a time function $f(t)$ whose spectrum $F(f)$ is limited to $-W \leq f \leq W$. The result of sampling the function at regular intervals spaced τ seconds apart is[10]

$$\hat{f}(t) = f(t)\sum_n \delta(t-n\tau) = \sum_n f(n\tau)\delta(t-n\tau). \tag{1}$$

The transform of

$$\sum \delta(t-n\tau) \text{ is } \sum \frac{1}{\tau}\,\delta\left(f-\frac{n}{\tau}\right).$$

Multiplication in the time domain corresponds to convolution in the frequency domain, and the first equality of (1) leads to

$$\hat{F}(f) = F(f) * \sum_n \frac{1}{\tau}\,\delta\left(f-\frac{n}{\tau}\right)$$
$$= \sum_n \frac{1}{\tau} F\left(f-\frac{n}{\tau}\right). \tag{2}$$

Apart from the weighting factor $1/\tau$, $\hat{F}(f)$ is seen to consist of replicas of $F(f)$ centered on the spectral lines $\delta(f-n/\tau)$, as illustrated in Fig. 1.[11] The possibility of recovering the original spectrum is insured if $1/\tau \geq 2W$; equality is permissible if $F(f)$ does not contain a δ-func-

[9] All time functions are assumed to be real unless specifically designated as being complex.

[10] All summations are from $-\infty$ to $+\infty$ unless otherwise stated.

[11] $F(f)$ is in general a complex function and is indicated symbolically in Fig. 1 (a). Weighting factors such as $1/\tau$ will be indicated as shown in Fig. 1 (b).

tion at $f=W$. Assuming that sampling takes place at the lowest permissible rate, one has $1/\tau=2W$. The original spectrum may be recovered by multiplying $\hat{F}(f)$ by the spectral window function $S(f)$ shown in Fig. 1(c). The equivalent operation in the time domain is the convolution of $f(t)$ by the inverse Fourier transform $s(t)$ of $S(f)$, *i.e.*,

$$f(t) = s(t) * \sum_n f(n\tau)\delta(t - n\tau) = \sum_n f(n\tau)s(t - n\tau).$$

Substituting $\tau = 1/2W$ and the functional form of $s(t)$,

$$f(t) = \sum_n f\left(\frac{n}{2W}\right) \frac{\sin 2\pi W\left(t - \dfrac{n}{2W}\right)}{2\pi W\left(t - \dfrac{n}{2W}\right)}. \tag{3}$$

The low-pass function $f(t)$ may also be subjected to second-order sampling. The two interlaced sampling trains

$$\sum_n \delta\left(t - \frac{n}{W}\right)$$

and

$$\sum_n \delta\left(t - \frac{n}{W} - \alpha\right)$$

will be designated by the letters A and B, respectively. The sampled functions are

$$f_A(t) = \sum_n f\left(\frac{n}{W}\right)\delta\left(t - \frac{n}{W}\right) \tag{4a}$$

and

$$f_B(t) = \sum_n f\left(\frac{n}{W} + \alpha\right)\delta\left(t - \frac{n}{W} - \alpha\right) \tag{4b}$$

and the corresponding spectra are given by

$$F_A(f) = F(f) * \sum_n W\delta(f - nW) \tag{5a}$$

$$F_B(f) = F(f) * \sum_n W\left(\frac{1}{\gamma}\right)^n \delta(f - nW) \tag{5b}$$

$$\gamma \equiv \exp\,(i2\pi\alpha W) \equiv \exp i\beta. \tag{5c}$$

The results of these convolutions are easily visualized: sketches of the spectra are shown in Fig. 2.[12] Since all time functions involved in this discussion are real, it suffices to consider their spectra for positive frequencies only. The spectral window functions $S_A(f)$ and $S_B(f)$ may be determined by the requirement

$$F_A(f)S_A(f) + F_B(f)S_B(f) = F(f), \quad 0 < f < W. \tag{6}$$

[12] Each spectrum is shown as the sum of two components which correspond to the convolutions of $F(f)$ with different spectral lines of the sampling function.

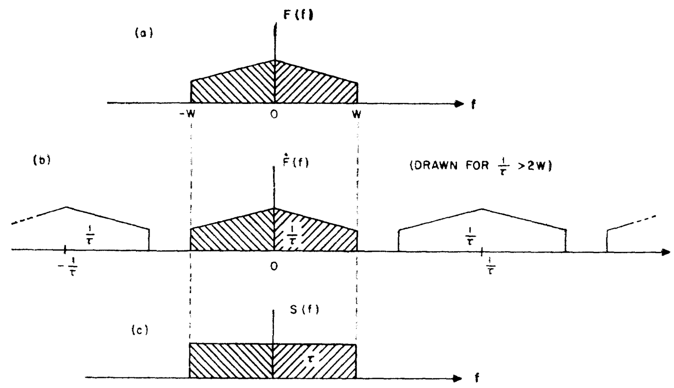

Fig. 1—First-order sampling of low-pass function.

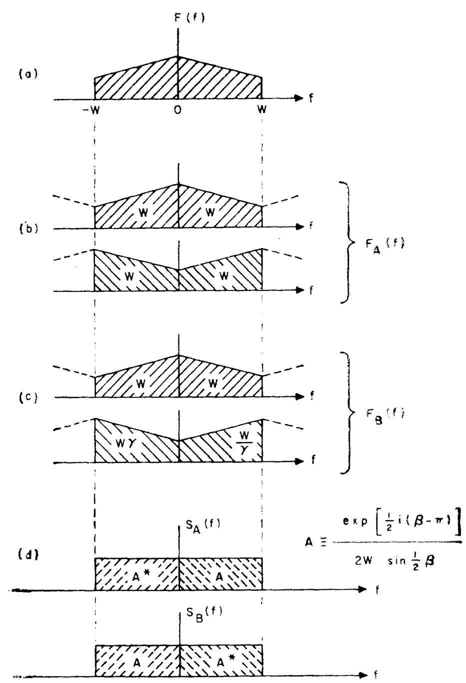

Fig. 2—Second-order sampling of low-pass function.

Inspection of Fig. 2 yields[13]

$$WS_A(f) + WS_B(f) = 1$$

$$WS_A(f) + \frac{W}{\gamma}S_B(f) = 0$$

whence

$$S_A(f) = S_B{}^*(f) = \frac{\exp\left[i\left(\dfrac{1}{2}\beta - \dfrac{\pi}{2}\right)\right]}{2W \sin \dfrac{1}{2}\beta}; \quad (0 < f < W).$$

[13] It is shown in Appendix I that these equations follow uniquely from (6).

Since the corresponding time functions are real, $S_{A,B}(-f) = S^*_{A,B}(f)$. The inverse Fourier transforms of $S_A(f)$ and $S_B(f)$ are the interpolating functions

$$s_A(t) = s_B(-t) = \frac{\cos(2\pi Wt - \pi\alpha W) - \cos \pi\alpha W}{2\pi Wt \sin \pi\alpha W} \cdot \quad (7a)$$

Finally,

$$\begin{aligned} f(t) &= s_A(t) * f_A(t) + s_B(t) * f_B(t) \\ &= \sum_n f\left(\frac{n}{W}\right) s_A\left(t - \frac{n}{W}\right) \\ &\quad + f\left(\frac{n}{W} + \alpha\right) s_A\left(-t + \frac{n}{W} + \alpha\right). \end{aligned} \quad (7b)$$

With $\alpha = 1/2W$, (7) reduces to (3).

Sampling of Band-Pass Functions

The spectrum is assumed to occupy the range $W_0 \leq |f| \leq (W_0 + W)$, as sketched in Fig. 3(a). In general, second-order sampling must be used,[14] and (4), (5) apply. The results of the convolutions are shown in Fig. 3(b) and 3(c). The spectral window functions $S_A(f)$ and $S_B(f)$ which are required to restore the original spectrum may be computed by a procedure similar to that leading to (6).[15] The result is indicated in Fig. 3(d). The corresponding interpolating functions are[16]

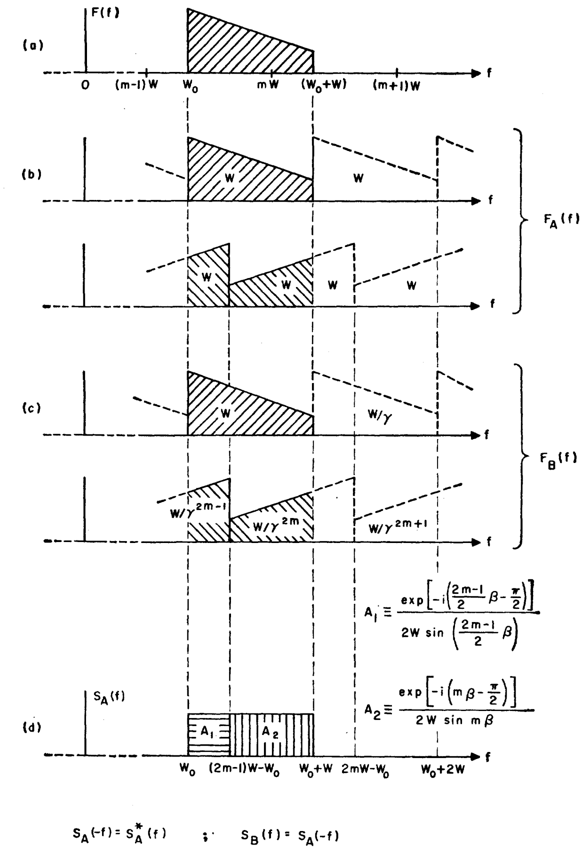

Fig. 3—Second-order sampling of band-pass function.

$$s_A(t) = \begin{cases} \dfrac{\cos[2\pi m\alpha W - 2\pi(W + W_0)t] - \cos[2\pi m\alpha W - 2\pi\{(2m-1)W - W_0\}t]}{2\pi Wt \sin 2\pi m\alpha W} \\ + \dfrac{\cos[(2m-1)\pi\alpha W - 2\pi\{(2m-1)W - W_0\}t] - \cos[(2m-1)\pi\alpha W - 2\pi W_0 t]}{2\pi Wt \sin[(2m-1)\pi\alpha W]} \end{cases}$$

$$s_B(t) = s_A(-t) \quad (8)$$

where m is the largest integer for which $(m-1)W < W_0$. Eq. (7b) applies provided that $s_A(t)$ is taken to be the function defined by (8). The separation α between the two interlaced sampling trains is arbitrary, except for the restriction that it may not be an integral multiple of $1/2W$ unless $W_0 = (m-1)W$. In the latter case, a development based on the first-order sampling of (1) and (2) yields the interpolation formula

$$f(t) = \sum_n f\left(\frac{n}{2W}\right) s\left(t - \frac{n}{2W}\right) \quad (9a)$$

$$s(t) = \frac{1}{2\pi Wt}[\sin 2\pi mWt - \sin 2\pi(m-1)Wt]. \quad (9b)$$

It is interesting to note that the repetitive nature of the spectra $F_A(f)$ and $F_B(f)$ of Fig. 3 offers the possibility of recovering not the original function but a frequency-translated version of it. For example, if the spectral window of Fig. 4 were used, the corresponding time function would represent an upward frequency translation of $f(t)$ by W cps.[17]

Quadrature and Hilbert Transform Sampling[18]

The sampling operation may be preceded by preparatory processing of the time function. The most obvious example is the representation of a band-pass function in terms of its in-phase and quadrature components, each of which may be sampled separately. Let

$$f(t) = A(t) \cos[\omega_0 t + \psi(t)] \quad (10)$$

[14] An exceptional case will be discussed at the end of this section.

[15] The only significant difference lies in the fact that the window functions must be computed separately for $W_0 < f < [(2m-1)W - 2W_0]$ and $[(2m-1)W - 2W_0] < f < (W_0 + W)$.

[16] This expression differs from (31) of Kohlenberg, *op. cit.*, only in notation; using $r \equiv 2m-1$. Kohlenberg's result is obtained.

[17] These remarks apply equally well to the low-pass function of Fig. 1. Amplitude modulation could have been achieved by the use of a suitable band-pass spectral window.

[18] Goldman, *op. cit.*

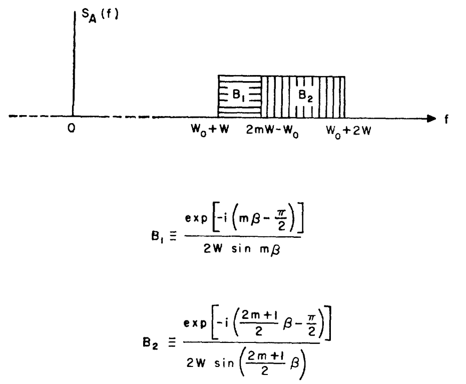

Fig. 4—Frequency-translation by use of spectral window.

and let its spectrum $F(f)$ be confined to a frequency band of width W, centered on f_0, as shown in Fig. 5. Providing that $f_0 > W$, the in-phase and quadrature components

$$f_I(t) = A(t)\cos\psi(t) \text{ and } f_Q(t) = A(t)\sin\psi(t) \quad (11)$$

may be obtained by multiplying $f(t)$ by $2\cos\omega_0 t$ and $-2\sin\omega_0 t$, respectively, and by filtering out the sum-frequency components. The corresponding spectra are given by

$$F_I(f) = \{F(f) * [\delta(f-f_0) + \delta(f+f_0)]\}_{lf}$$

$$F_Q(f) = \{F(f) * i[\delta(f-f_0) - \delta(f+f_0)]\}_{lf} \quad (12)$$

where the subscript lf indicates that the sum-frequency components have been discarded. These relations are illustrated in Fig. 5. Since $f_I(t)$ and $f_Q(t)$ are band-limited to $-W/2 \leq f \leq W/2$, each may be sampled at the rate of W samples per second. Reconstruction of the original function involves separate interpolations of $f_I(t)$ and $f_Q(t)$, multiplication by $\cos\omega_0 t$ and $\sin\omega_0 t$, respectively, and addition of the results.

First-order sampling of a band-pass function $f(t)$ and of its Hilbert transform

$$f_H(t) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(\tau)d\tau}{t-\tau} = f(t) * \left(-\frac{1}{\pi t}\right) \quad (13)$$

suffices to determine the function. This result is readily obtained by observing that the spectrum $F_H(f)$ of $f_H(t)$ is given by

$$F_H(f) = F(f)\mathfrak{F}\left\{-\frac{1}{\pi t}\right\} = F(f)[-i \operatorname{sgn} f]$$

where $F(f)$ is the spectrum of $f(t)$ and is assumed to be limited to the band $W_0 \leq |f| \leq (W_0+W)$. The functions $f(t)$ and $f_H(t)$ are now sampled at a rate of W times per second. Using the results of Fig. 3(b), the periodic spectra $\hat{F}(f)$ and $\hat{F}_H(f)$ may be sketched immediately, as

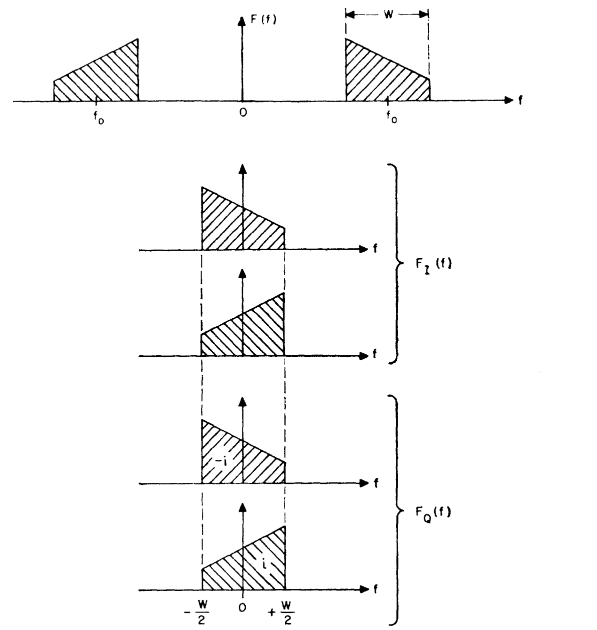

Fig. 5—Quadrature sampling. The direction of cross-hatching distinguishes the positive- and negative-frequency parts of $F(f)$ and the spectral contributions derived from them.

shown in Fig. 6.[19] The required window functions $S(f)$ and $S_H(f)$ may be determined by inspection [Fig. 6(d)], and the corresponding interpolating functions are

$$s(t) = \frac{\sin\pi Wt}{\pi Wt}\cos 2\pi\left(W_0 + \frac{W}{2}\right)t \quad (14a)$$

$$s_H(t) = -\frac{\sin\pi Wt}{\pi Wt}\sin 2\pi\left(W_0 + \frac{W}{2}t\right). \quad (14b)$$

These two functions are Hilbert transforms, as anticipated in the notation. Finally,

$$f(t) = \sum_n f\left(\frac{n}{W}\right)s\left(t-\frac{n}{W}\right) + f_H\left(\frac{n}{W}\right)s_H\left(t-\frac{n}{W}\right). \quad (15)$$

Sampling of Periodic Functions[20]

While the preceding discussion does not exclude line spectra, its results are not particularly useful for periodic functions since the interpolation process is based on an infinite number of samples rather than a finite number of points within one period. The necessary modifications will be outlined for the low-pass case.

Let $f(t)$ be a periodic function of period T, which contains no spectral components above the Nth harmonic, and let the function be sampled at intervals of τ seconds. Fig. 1 applies with $W = N/T$. The inequality $1/\tau > 2W = 2N/T$ cannot be satisfied with the equal sign since this choice would destroy the identity of the spectral line at $f = N/T$. The lowest acceptable rate of equispaced sampling is therefore given by $\tau = T/(2N+1)$. The re-

[19] The similarity between Figs. 5 and 6 is evident. These sketches illustrate the close connection between quadrature sampling and the present procedure.
[20] Goldman, *op. cit.*

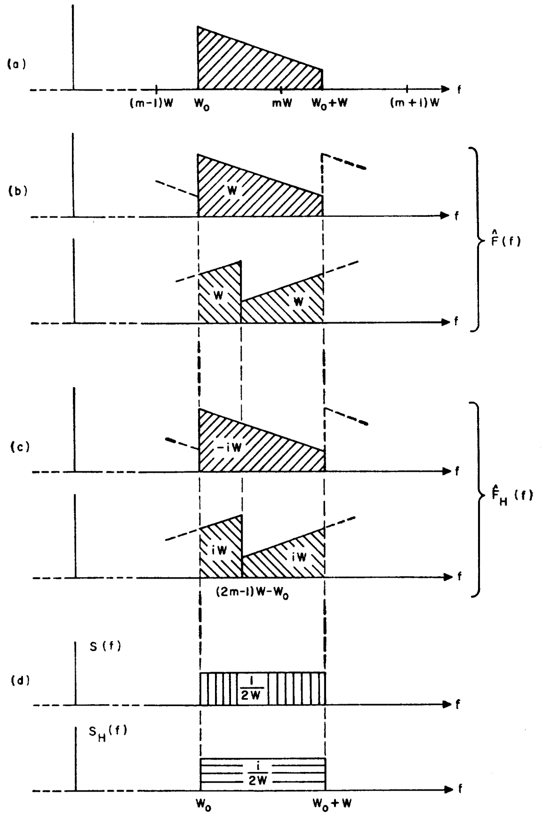

Fig. 6.—Hilbert transform sampling.

sulting spectrum is sketched in Fig. 7(a). Since there are gaps between successive replicas of $F(f)$, the spectral window is not uniquely determined. The window function $S(f)$ shown in Fig. 7(b) has the advantage of providing independent sampling since the corresponding interpolating function $s(t)$ has zeros at all sampling points but one. Eq. (3) may now be applied with obvious changes of notation:

$$f(t) = \sum_{n=-\infty}^{+\infty} f(n\tau)s(t - n\tau); \qquad \tau = \frac{T}{2N+1}, \tag{16a}$$

$$s(t) = \frac{\sin 2\pi\left(\frac{N+\frac{1}{2}}{T}\right)t}{2\pi\left(\frac{N+\frac{1}{2}}{T}\right)t}. \tag{16b}$$

Since $f(t)$ is periodic with period T, (16a) may be written as

$$f(t) = \sum_{n=0}^{2N} f(n\tau)p(t - n\tau)$$

where

$$p(t) = \sum_{k=-\infty}^{+\infty} s(t - kT) = \frac{\sin (2N+1)\frac{\pi}{T}t}{(2N+1)\sin\frac{\pi}{T}t}. \tag{17}$$

The last equality is proved in Appendix II.

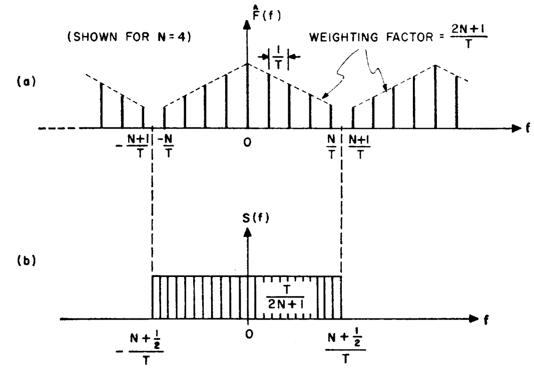

Fig. 7—Sampling of periodic low-pass function.

Sampling of a Function and Its Derivative

Simultaneous sampling of a function and of its derivative yields two periodic spectra from which the original spectrum may be recovered by appropriate spectral windows. It is assumed that the spectrum of $(d/dt)f(t)$ is given by $i2\pi fF(f)$. The procedure will be illustrated for band-limited, low-pass functions. Writing $F_+(f)$ and $F(f)$ for the positive and negative frequency parts of $F(f)$, the spectra of $f(t)$ and $f'(t)$ are sketched[21] in Fig. 8(a). The spectra of the sampled functions

$$f_A(t) = f(t)\sum_n \delta\left(t - \frac{n}{W}\right)$$

and

$$f_B(t) = f'(t)\sum_n \delta\left(t - \frac{n}{W}\right) \tag{18}$$

are shown in Fig. 8(b). Using the condition of (6), one obtains in the range $0<f<W$,

$$\begin{aligned} &WF_+(f)S_A(f) + i2\pi fWF_+(f)S_B(f) = F_+(f) \\ &WF_-(f-W)S_A(f) \\ &\qquad + i2\pi(f-W)WF_-(f-W)S_B(f) = 0 \end{aligned} \tag{19}$$

whence

$$\begin{aligned} S_A(f) &= \frac{1}{W}\left(1 - \frac{f}{W}\right), \qquad 0<f<W \\ S_B(f) &= \frac{1}{i2\pi W^2}, \qquad 0<f<W. \end{aligned} \tag{20}$$

The interpolating functions are

$$\begin{aligned} s_A(t) &= \left(\frac{\sin \pi Wt}{\pi Wt}\right)^2 \\ s_B(t) &= ts_A(t) \end{aligned} \tag{21}$$

so that

$$\begin{aligned} f(t) &= f_A(t) * s_A(t) + f_B(t) * s_B(t) \\ &= \sum_n \left[f\left(\frac{n}{W}\right) + \left(t - \frac{n}{W}\right)f'\left(\frac{n}{W}\right)\right]s_A\left(t - \frac{n}{W}\right). \end{aligned} \tag{22}$$

[21] These sketches are equivalent to (8) of Fogel, *op. cit.*

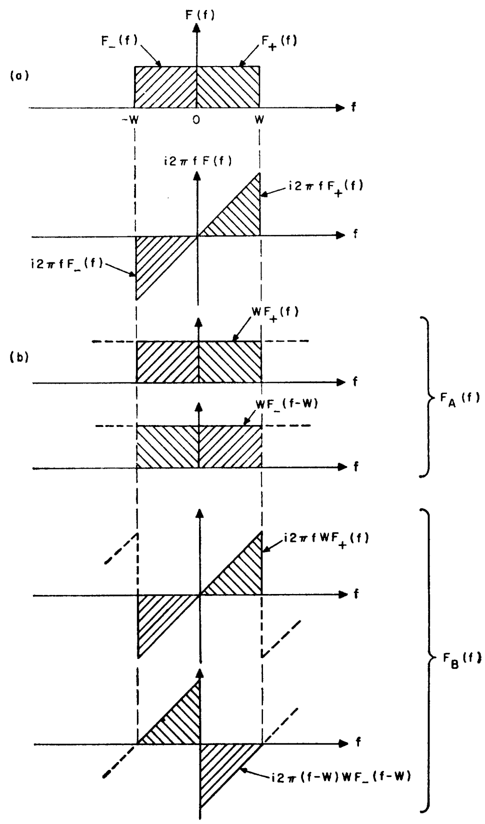

Fig. 8—Sampling of a function and its derivative. The direction of cross-hatching distinguishes the positive- and negative-frequency parts of $F(f)$ and the spectral contributions derived from them.

The more general case of first-order sampling of a real low-pass function and its first R derivatives may be treated by similar methods. The derivation is straightforward, but somewhat lengthy; it is given in Appendix III and leads to the following results. The function and its R derivatives are sampled at intervals of $\tau \equiv (R+1)/2W$ seconds. The spectral window function $S^{(r)}(f)$ for the rth derivative ($r=0, 1, \cdots, R$) is obtained in $2(R+1)$ segments, each of width $W/(R+1)$, starting at $f=-W$:

$$S^{(r)}(f) = \sum_{n=-(R+1)}^{R} S_n^{(r)}(f).$$

Each segment represents a separate problem; however the following relations reduce the number of functions which must be determined:

$$\begin{aligned} S_{-(n+1)}^{(r)}(-f) &= [S_n^{(r)}(f)]^* \\ S_{2m+1}^{(r)}(f) &= S_{2m}^{(r)}(f) \qquad (R \text{ odd}) \\ S_{2m+1}^{(r)}(f) &= S_{2m+2}^{(r)}(f) \qquad (R \text{ even}). \end{aligned}$$

The $S_n^{(r)}(f)$ for the remaining $(R+1)/2$ (R odd) or $(R+2)/2$ (R even) values of n are found by solving the following sets[22] of equations:

[22] Each set consists of $(R+1)$ equations, corresponding to the $(R+1)$ unknown functions $S_n^{(0)}, \cdots, S_n^{(R)}$.

$$\begin{aligned} &\sum_{r=0}^{R} S_n^{(r)}(f)[i2\pi(f-k/\tau)]^r = \tau\delta_{0,k}, \qquad (f>0) \\ &k = k_{\min}(n), \cdots, [k_{\min}(n)+R] \\ &n = 0, 2, 4, \cdots, (R-1) \qquad (R \text{ odd}) \\ & = 0, 2, 4, \cdots, R \qquad (R \text{ even}) \end{aligned}$$

where $\delta_{0,k}$ is one or zero according as k is zero or nonzero, and where $k_{\min}(n)$ is the smallest integer such that

$$k_{\min}(n) \geq \frac{n-R}{2}.$$

The interpolating functions $s^{(r)}(t)$ are then obtained as the inverse Fourier transforms of the $S^{(r)}(f)$, and

$$\begin{aligned} f(t) &= \sum_{r=0}^{R} \sum_m \frac{d^r f(m\tau)}{dt^r} s^{(r)}(t-m\tau) \\ &= \sum_m \left[\sum_{r=0}^{R} \frac{d^r f(m\tau)}{dt^r} s^{(r)}(t-m\tau) \right]. \end{aligned}$$

Appendix I

If the positive and negative-frequency parts of $F(f)$ are designated as $F_+(f)$ and $F_-(f)$, [where $F_+^*(-f) = F_-(f)$], one has in the interval $0<f<W$

$$\begin{aligned} F_A(f) &= WF_+(f) + WF_-(f-W), \\ F_B(f) &= WF_+(f) + \frac{W}{\gamma} F_-(f-W). \end{aligned}$$

Substituting into (6),

$$F_+(f)[WS_A(f) + WS_B(f) - 1] + F_-(f-W)\left[WS_A(f) + \frac{W}{\gamma} S_B(f)\right] = 0.$$

There is, in general, no functional relationship between $F_+(f)$ and $F_-(f-W) = F_+^*(W-f)$; equating to zero the coefficients of $F_+(f)$ and $F_-(f-W)$, one obtains the two equations following (6).

Appendix II

$$p(t) = \sum_{k=-\infty}^{\infty} s(t-kT) = s(t) * \left[\sum_{k=-\infty}^{\infty} \delta(t-kT)\right].$$

The corresponding spectrum is

$$\begin{aligned} P(f) &= S(f) \sum_{k=-\infty}^{\infty} \frac{1}{T} \delta\left(f - \frac{k}{T}\right) \\ &= \frac{1}{2N+1} \sum_{k=-N}^{N} \delta\left(f - \frac{k}{T}\right). \end{aligned}$$

The last equality may be verified by inspection of the window function $S(f)$ shown in Fig. 7(b). Finally,

$$p(t) = \frac{1}{2N+1} \sum_{k=-N}^{N} (e^{i(2\pi/T)})^k = \frac{\sin (2N+1) \dfrac{\pi t}{T}}{(2N+1) \sin \dfrac{\pi t}{T}}.$$

Appendix III

It will be assumed that the spectrum of the rth derivative is $(i2\pi f)^r F(f)$. The function and its first R derivatives are sampled at intervals of $\tau \equiv (R+1)/2W$ seconds. Their spectra are therefore convolved with the impulse function train

$$\frac{1}{\tau} \sum_k \delta\left(f - \frac{k}{\tau}\right). \tag{23}$$

Each of the $(R+1)$ spectra extends from $-W$ to W, and will be divided into $2(R+1)$ intervals of width $W/R+1 = 1/2\tau$, starting at $f = -W$. Let $F_n(f)$ be equal to $F(f)$ in the nth interval and zero outside it, *i.e.*,

$$F(f) = \sum_{n=-(R+1)}^{R} F_n(f). \tag{24}$$

Since $f(t)$ is assumed to be real, $F_{-(n+1)}(-f) = F_n^*(f)$. Fig. 9 shows the spectrum $F(f)$, the numbering of its $(R+1)$ intervals, and the convolving train of impulse functions.

The convolution process is visualized in terms of erecting replicas centered on the impulse functions. It is easily seen that a replica of $F(f)$, centered on the impulse function at $f = k/\tau$, will contribute to the $(2k+j)$th interval the function

$$\frac{1}{\tau} F_j\left(f - \frac{k}{\tau}\right).$$

Using the notation

$$D^k[g(f)] \equiv g\left(f - \frac{k}{\tau}\right),$$

a replica of $F(f)$ centered on $\delta(f - k/\tau)$ will contribute to the nth interval the function $1/\tau D^k[F_{n-2k}(f)]$. Let $F^{(r)}(f)$ be the spectrum obtained from the convolution of $(i2\pi f)^r F(f)$ with the train on impulse functions of (23), and let $F_n^{(r)}(f)$ be its nth segment, *i.e.*,

$$F^{(r)}(f) = \sum_{n=-(R+1)}^{R} F_n^{(r)}(f).$$

Then it follows from the preceding discussion that[23]

$$F_n^{(r)}(f) = \frac{1}{\tau} \sum_{k=k_{\min}(n)}^{k_{\min}(n)+R} D^k[(i2\pi f)^r F_{n-2k}(f)]. \tag{25}$$

Since $F_{n-2k}(f)$ vanishes outside the interval $(-W, W)$,

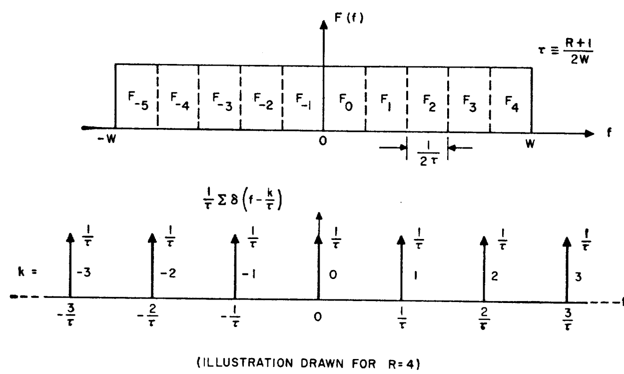

Fig. 9— Sampling of a function and its first R derivatives.

the summation may be restricted to those integral values of k which satisfy the inequality

$$-(R+1) \leq (n - 2k) \leq R.$$

For each n, there are therefore $(R+1)$ values of k, starting with $k_{\min}(n)$; the latter is the smallest integer which satisfies

$$k_{\min}(n) \geq \frac{n-R}{2}. \tag{26}$$

In order to recover $F(f)$ from the $(R+1)$ spectra $F^{(r)}(f)$, each $F^{(r)}(f)$ is multiplied by a spectral window function $S^{(r)}(f)$. One then demands that

$$\sum_{r=0}^{R} S^{(r)}(f) F^{(r)}(f) = F(f). \tag{27}$$

Since $f(t)$ was assumed to be real, the $S^{(r)}(f)$ are spectra of real functions, and it suffices to consider positive frequencies only. Each of the $(R+1)$ positive-frequency intervals must be considered separately so that (27) represents $(R+1)$ separate equations;

$$\sum_{r=0}^{R} S_n^{(r)}(f) F_n^{(r)}(f) = F_n(f); \quad n = 0, 1, \cdots R \tag{28}$$

where $S_n^{(r)}(f)$ represents the nth segment of $S^{(r)}(f)$, with

$$S_{-(n+1)}^{(r)}(-f) = [S_n^{(r)}(f)]^*. \tag{29}$$

Substituting (25) into (28),

$$\sum_{r=0}^{R} S_n^{(r)}(f) \frac{1}{\tau} \sum_{k=k_{\min}(n)}^{k_{\min}(n)+R} D^k[(2\pi f i)^r F_{n-2k}(f)] = F_n(f)$$
$$n = 0, 1, \cdots, R. \tag{30}$$

Interchanging orders of summation,

$$\sum_{k=k_{\min}(n)}^{k_{\min}(n)+R} D^k[F_{n-2k}(f)] \sum_{r=0}^{R} S_n^{(r)}(f) D^k[(2\pi f i)^r]$$
$$= \tau F_n(f). \tag{31}$$

Since the $F_n(f)$ are independent, the coefficient of each $D^k[F_{n-2k}]$ must be identically zero. For each value of n, (31) thus provides $(R+1)$ equations

[23] Eq. (25) is equivalent to (14) of Fogel, *op. cit.*

$$\sum_{r=0}^{R} S_n^{(r)}(f) D^k[(2\pi f i)^r] = \tau \delta_{0,k}$$
$$k = k_{\min}(n), \cdots, [k_{\min}(n) + R]$$
$$n = 0, 2, \cdots, R \tag{32}$$

where $\delta_{0,k}$ is one or zero according as k is zero or nonzero.

Inspection of (26) shows that for odd R,

$$k_{\min}(0) = k_{\min}(1), \qquad k_{\min}(2) = k_{\min}(3), \text{ etc.},$$

while for even R,

$$k_{\min}(1) = k_{\min}(2), \qquad k_{\min}(3) = k_{\min}(4), \text{ etc.}$$

Thus

$$S_{2m+1}^{(r)} = S_{2m}^{(r)} \qquad (R \text{ odd})$$
$$S_{2m+1}^{(r)} = S_{2m+2}^{(r)} \qquad (R \text{ even}).$$

It is therefore sufficient to solve (32) for even values of n so that there are $(R+1)/2$ or $(R+2)/2$ sets of equations, according to whether R is odd or even.

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An Application of Piecewise Approximations to Reliability and Statistical Design*

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Summary—If a random variable can be expressed as a weighted sum of other random variables having known distributions which can be approximated piecewise by, for example, polynomials, the distribution of the random variable can be obtained, relatively easily, by the use of the algorithm described in this paper.

Introduction

In many systems, such as missile, computer, or control systems, there may arise a need for the determination of the probability of failure due to the gradual deterioration of the system components. Associated with this need is the determination of the probability that a specified characteristic of the system or a part of the system will be outside of acceptable limits on account of a chance unfavorable combination of component values. Examples of specific characteristics might be: the delay of a pulse circuit, the phase margin in a feedback control system, the gain of a linear amplifier—quantities all of which are functions of the values of the components involved such as resistances, capacitances, vacuum tube transconductances, and plate resistances, etc. Denote the characteristic by T and the values of the components involved by $x_1, x_2, \cdots, x_n$. Then

$$T = T(x_1, x_2, \cdots, x_n). \tag{1}$$

It is often possible to express sufficiently accurately the deviation δT of the characteristic T from some nominal value in terms of the deviations of the component values, δx_i, from their mean values as follows:

$$\delta T = a_1 \delta x_1 + a_2 \delta x_2 + \cdots + a_n \delta x_n. \tag{2}$$

The numbers, $a_1, a_2, \cdots, a_n$ can be determined either by experiment or by calculation. Eq. (2) may be rewritten:

$$\delta T/T_0 = b_1 \delta x_1/x_{10} + b_2 \delta x_2/x_{20} + \cdots + b_n \delta x_n/x_{n0};$$
$$b_i = a_i x_{i0}/T_0 \qquad i = 1, 2, \cdots n \tag{3}$$

where $T_0, x_{10}, x_{20}, \cdots, x_{n0}$ are the "mean" values of $T, x_1, \cdots, x_n$. $[T_0 \approx T(x_{10}, x_{20}, \cdots, x_{n0})]$. Eq. (3) can be considered as expressing the percentage change in the characteristic resulting from certain percentage changes in the components involved, as the equality is not affected by multiplying both sides by 100. The problem then becomes one of determining how ξ is distributed knowing how the ξ_i are distributed where

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \tag{4}$$

and $\xi = \delta T/T_0$, $\xi_i = b_i \delta x_i / x_{i0}$; $i = 1, 2, \cdots, n$, the mean of ξ_i is zero for $i = 1, 2, \cdots, n$, and the mean of ξ is zero. The ξ_i are assumed to be independent random variables.[1]

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[1] The assumption that the means of ξ and ξ_i are zero is not necessary, but simplifies the discussion that follows.